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SKEW POLYNOMIAL RINGS AND OVERRINGS.

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Skew Polynomial Rings and Overrings.

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Summary.

Given a ring R and a monomorphism $\alpha: R \rightarrow R$, it is possible to construct a minimal overring $A(R, \alpha)$ of R to which α extends as an automorphism - this was done by D.A. Jordan in [16]. Chapter 1 presents this construction, and the remainder of the thesis is devoted to the study of the ring $A(R, \alpha)$ and its applications.

Chapter 2 deals with the ideal structure of $A(R, \alpha)$: the prime, semi-prime and nilpotent ideals are examined, and it is shown that if nil left ideals of R are nilpotent, then the nilpotent radical of $A(R, \alpha)$ is nilpotent. It is also shown that if R has finite left Goldie dimension n , then the left Goldie dimension of $A(R, \alpha)$ cannot exceed n - however, an example is constructed to show that the ascending chain condition on left annihilators need not be passed from R to $A(R, \alpha)$.

In chapter 3, several aspects of $A(R, \alpha)$ are studied under the assumption that it is left Noetherian, and a question raised by Jordan in [16] is settled by an example where R is a ring of Krull dimension 1, but $A(R, \alpha)$ does not have Krull dimension. Examination of the Jacobson radical of $A(R, \alpha)$, and a proof of the fact that maximal left ideals of left Artinian rings are closed, then leads to a generalization of a result of Jategaonkar, which states that if R is left Artinian, then $\alpha^{-1}(J(R)) = J(R)$.

Chapter 4 first finds a condition on R equivalent to $A(R, \alpha)$ being a full quotient ring, and then finds a regularity condition on R which is equivalent to $A(R, \alpha)$ having a left Artinian left quotient ring in the case where R is left Noetherian with an α -invariant nilpotent radical.

Finally, $A(R, \alpha)$ is applied to the skew Laurent polynomial ring $R[x, x^{-1}, \alpha]$ where α is a monomorphism, to obtain sufficient conditions for $R[x, x^{-1}, \alpha]$ to be semiprimitive, primitive, and Jacobson. Also, equivalent conditions on R are found for $R[x, x^{-1}, \alpha]$ to be simple.

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Conventions.

All rings will be assumed to have unity, but need not be commutative. All modules over a ring R will be taken to be left R -modules unless otherwise stated - the notation ${}_R M$ will signify that M is to be regarded as a left R -module, while M_R is the corresponding right-handed notation.

The letter α will always denote an injective ring endomorphism such that $\alpha(1) = 1$; α will not be surjective unless specified otherwise. If $S \subseteq R$ then $\alpha^{-1}(S)$ will denote the set $\{r \in R \mid \alpha(r) \in S\}$.

The abbreviations acc and dcc stand for ascending chain condition and descending chain condition respectively; " $\dim M$ " will mean the Goldie dimension of the module M .

\mathbb{N} will denote the natural numbers $\{1, 2, 3, \dots\}$, \mathbb{N}_0 will denote the set $\mathbb{N} \cup \{0\}$, and \mathbb{Z} will denote the integers.

If $f: R \rightarrow T$ is a function and $S \subseteq R$ then the restriction of f to S will be denoted by $f|_S$.

If p is a property such that "left p " is not the same as "right p ", then to say that " A is p " will mean A is both left p and right p .

INTRODUCTION

In this thesis, the general situation we shall be concerned with is that of a ring R , with a ring monomorphism $\alpha: R \rightarrow R$, which will not be assumed to be surjective.

The fact that α is not surjective makes it very difficult to study the effect that it has on ring-theoretic properties of R : some problems, which may be solved immediately in the case where α is an automorphism, suddenly assume much larger proportions when α is not surjective.

For instance, it is clear that if α is an automorphism, and $r \in R$ is regular, then $\alpha(r)$ is regular. This is, in general, false when α is assumed only to be a monomorphism, although it is known (see [13]) to be true provided that R possesses a left Artinian left quotient ring. Another striking deviation from the automorphism case is that, given a left ideal I of R , $\alpha(I)$ need not be a left ideal.

It is evident, therefore, that automorphisms are much easier to handle than monomorphisms, and it is with this in mind that D.A. Jordan [16] constructs a minimal overring, denoted $A(R, \alpha)$, of R , to which the monomorphism α extends as an automorphism. This construction, along with the important elementary facts about $A(R, \alpha)$ as proved by Jordan, is presented in chapter 1.

Chapters 2, 3 and 4 are all concerned with the study of the ring $A(R, \alpha)$, and how its properties are related to the properties of R .

In some ways, $A(R, \alpha)$ is well behaved, but in other ways it can be extremely difficult to handle - for instance, as will be seen in chapter 1, if R is left Artinian, then so is $A(R, \alpha)$, but on the other hand, there are easy examples which show that it is possible for R to be left Noetherian, but not $A(R, \alpha)$.

Chapter 2 deals with several ideal-theoretic questions about $A(R, \alpha)$ which were not considered in [16] - here it is shown that if R has finite left Goldie dimension, then so does $A(R, \alpha)$, but an example is produced to show that the ascending chain condition on left annihilators does not persist on passage from R to $A(R, \alpha)$.

Although, as noted above, the left Noetherian condition can often fail in $A(R, \alpha)$ even when R is left Noetherian, in [16], Jordan found conditions on R which are equivalent to $A(R, \alpha)$ being left Noetherian. It is the purpose of chapter 3 to study $A(R, \alpha)$ under these assumptions.

Chapter 4 examines two aspects of the quotient ring problem for $A(R, \alpha)$, with the object firstly of finding conditions on R equivalent to $A(R, \alpha)$ being a full quotient ring, and then of discussing when $A(R, \alpha)$ has a left Artinian left quotient ring.

Part of the underlying approach to chapters 2 and 4 is that every attempt is made to get around the bad behaviour of $A(R, \alpha)$ - thus, rather than assuming that $A(R, \alpha)$ is (for example) Noetherian, it is preferred to assume that R is Noetherian, and then to work towards the

same result. The best examples of how this approach can work are the study of the nilpotent radical of $A(R, \alpha)$, in chapter 2, and the part of chapter 4 dealing with left Artinian left quotient rings. Unfortunately, this ideal cannot always be attained, and it will occasionally be necessary to assume that $A(R, \alpha)$ is left Noetherian.

An interesting application of $A(R, \alpha)$ occurs in the area of skew Laurent polynomial rings. These have been studied (in the case where α is an automorphism) by Jordan in [15] and [17] and by Goldie and Michler [7], who showed that $R[x, x^{-1}, \alpha]$ could be used to solve certain problems in group rings. The applications of $A(R, \alpha)$ depend upon the observation that, given a monomorphism on a ring R , $R[x, x^{-1}, \alpha] = A(R, \alpha)[x, x^{-1}, \alpha]$, where the α appearing on the right is an automorphism of $A(R, \alpha)$.

Thus, once the relationship between R and $A(R, \alpha)$ is sufficiently understood, the existing theorems about $R[x, x^{-1}, \alpha]$, where α is an automorphism, may be extended to include the case where α is only a monomorphism. This is done in chapter 5 with the results in [15] and [17] on primitivity and simplicity of $R[x, x^{-1}, \alpha]$.

CHAPTER 1

PRELIMINARIES.

This chapter contains virtually all the known results which will be needed in the work ahead. First, ring-theoretical generalities, such as terminology and localization, are discussed, followed by two more specialized topics: the skew Laurent polynomial ring $R[x, x^{-1}, \alpha]$ and the overring $A(R, \alpha)$ - the central object of the thesis.

In general, proofs will only be included if they are not readily accessible in the literature.

§1. Terminology.

In this section, general ring-theoretic terminology is standardized, and will apply throughout.

A subring S of a ring R is said to be nil if each element of S is nilpotent, i.e. if for all $x \in S$, there exists $n \in \mathbb{N}$ with $x^n = 0$. S is said to be nilpotent if there exists $n \in \mathbb{N}$ with $S^n = 0$, and is said to have index of nilpotence n if $S^n = 0 \neq S^{n-1}$.

The nilpotent radical $N(R)$ of R is the sum of all the nilpotent left ideals of R ; it also coincides with the sum of all the nilpotent right, or two-sided, ideals of R . Although in general $N(R)$ need not itself be nilpotent, certain chain conditions may be imposed to ensure that it is - for instance, Levitzki's theorem (corollary 1.8 of [3]), states that if R is left Noetherian then $N(R)$ is nilpotent.

If $N(R) = 0$, i.e. R has no non-zero nilpotent one-sided ideals, then R is called semiprime; R will be called prime if for all ideals A and B of R , $AB = 0$ implies that either $A = 0$ or $B = 0$. An ideal I of R will be called prime if R/I is a prime ring, and I will be called semiprime if R/I is a semiprime ring.

The Jacobson radical, $J(R)$, is the intersection of all the maximal left ideals of R , and if $J(R) = 0$ then R is called semi-primitive. It should be noted that $J(R)$ coincides with the intersection of all the maximal right ideals of R .

The centre, $C(R)$, is the subring of R given by $C(R) = \{r \in R \mid sr = rs \text{ for all } s \in R\}$.

An element c of R is said to be left regular if $rc = 0$ implies $r = 0$, right regular if $cr = 0$ implies $r = 0$, and regular if it is both left and right regular.

If I is an ideal of R then $c \in R$ is (left or right) regular modulo I if $r+I$ is (left or right) regular in the factor ring R/I . The set of regular elements of a ring R is denoted by $C_R(0)$, and the set of regular elements modulo I is denoted by $C_R(I)$. The symbol ${}^lC_R(I)$ (resp. ${}^rC_R(I)$) will be used to denote the left (resp. right) regular elements modulo I , and the subscript R may often be omitted in situations where there is no ambiguity.

An R -module M is said to be Artinian if it has descending chain condition on submodules; M is said to be Noetherian if it has ascending chain condition on submodules. Thus, a ring R is called left Artinian if it has descending chain condition on left ideals, and the terms right Artinian, left Noetherian and right Noetherian are defined similarly for R .

If R is a Noetherian (i.e., left and right Noetherian) ring, then a left ideal I of R is said to be an Artinian left ideal if I is Artinian as a left R -module. The sum of all such left ideals, denoted A , is itself an Artinian left ideal. In fact, A is an ideal, and by Lenagan's lemma, ([3], corollary 4.2) which states that an ideal of a Noetherian ring is Artinian as a left ideal iff it is Artinian as a right ideal, A is the unique largest Artinian left or right ideal of R . It is called the Artinian radical of R .

An R -module M is said to have finite Goldie dimension if there does not exist an infinite direct sum of non-zero submodules of M ; M is said to be uniform if it is non-zero and any two non-zero submodules of M have non-zero intersection. A submodule E of M is said to be essential if E has non-zero intersection with each non-zero submodule of M .

If M is a module of finite Goldie dimension then the maximal length of a direct sum of uniform submodules is well-defined (see lemma 1.9 (b) of [3]), and is called the Goldie dimension of M .

If $S \subseteq R$ then the set $\ell(S) = \{x \in R \mid xS = 0\}$ is a left ideal of R , called the left annihilator of S . A left ideal of the form $\ell(S)$ will be called an annihilator left ideal. Right annihilators and annihilator right ideals are defined similarly.

A ring R is called a left Goldie ring if R has finite left Goldie dimension (i.e., ${}_R R$ has finite Goldie dimension) and R has ascending chain condition on annihilator left ideals.

Finally, a ring R is said to be left primitive if it has an irreducible faithful left R -module, and an ideal I of R is said to be left primitive if R/I is a left primitive ring.

§2. Localization and Quotient Rings.

Let S be a multiplicatively closed subset of R consisting of regular elements.

Then a left localization of R at S is an overring $S^{-1}R$ of R such that

- (i) each element of S is a unit of $S^{-1}R$;
- (ii) each element of $S^{-1}R$ can be written in the form $s^{-1}r$,
where $s \in S$ and $r \in R$.

It is well-known (see for example, p.118-119 of [12]) that $S^{-1}R$ exists iff S is a left Ore subset of R - i.e., for each $s \in S$ and $r \in R$, there exists $s_1 \in S$, $r_1 \in R$ such that $s_1 r = r_1 s$.

If S is the set of all regular elements of R , and if S is a left Ore subset, then $S^{-1}R$ is called the left quotient ring of R .

In general, a ring is called a quotient ring if each of its regular elements is a unit. Note that if Q is the left quotient ring of a ring R , then Q is a quotient ring.

The following result is standard, and will be referred to later on.

1.1 Theorem:

Let R be a ring, $S \subseteq R$ a multiplicatively closed left Ore subset of regular elements. If Q and T are left localizations of R at S , then Q is isomorphic to T .

Proof:

See p.119 of [12].

1.2 Definition:

(i) A prime ideal P of R is said to be left localizable if $C(P)$ is a left Ore subset of R .

(ii) A ring R is said to be local if $R/J(R)$ is simple Artinian.

The following theorem is standard.

1.3 Theorem:

Let R be a prime, left Noetherian ring, P a left localizable prime ideal. Then $C(P) \subseteq C(0)$, and if R_P denotes the left localization of R at $C(P)$, then R_P is a local ring with maximal ideal $R_P P$.

Proof:

Denote by I the set $\{r \in R \mid cr = 0 \text{ for some } c \in C(P)\}$. Then, using the fact that $C(P)$ is a left Ore subset of R , it can be shown that I is an ideal of R . Since R is prime, if I is non-zero then it is essential as a left ideal, so by Goldie's theorem (theorem 1.10 of [3]), I contains a regular element. This is clearly impossible, so $I = 0$ - therefore $C(P) \subseteq C'(0)$. To see that each element of $C(P)$ is also left regular, let $c \in C(P)$ and consider the chain $\ell(c) \subseteq \ell(c^2) \subseteq \dots \subseteq \ell(c^n) \subseteq \dots$. Since R is left Noetherian, there must exist $k \geq 1$ with $\ell(c^k) = \ell(c^{k+1})$. Let $r \in R$

be such that $rc = 0$.

Since $C(P)$ is a left Ore subset of R , there exists $r_1 \in R$ and $c_1 \in C(P)$ such that $r_1 c^k = c_1 r$. Thus, $r_1 c^{k+1} = 0$, and $r_1 \in \ell(c^{k+1}) = \ell(c^k)$. Therefore $c_1 r = 0$, and since $c_1 \in C(P) \subseteq C'(0)$ (from above), $r = 0$, and c is left regular. Therefore $C(P) \subseteq C(0)$.

Now consider R_p . Since R_p is left Noetherian, the increasing chain $R_p P c^{-1} \subseteq R_p P c^{-2} \subseteq \dots$ of left ideals must terminate, for any $c \in C(P)$. Therefore, for some $k \geq 1$, $R_p P c^{-k} = R_p P c^{-(k+1)}$, and hence $R_p P = R_p P c^{-1}$. Thus $R_p P$ is an ideal of R_p .

In fact $R_p P$ is a quasi-regular ideal, since for any $c^{-1}a \in R_p P$ (where $c \in C(P)$, $a \in P$), $1 - c^{-1}a = c^{-1}(c-a)$, and both c^{-1} and $c-a$ are units of R_p . Therefore $R_p P \subseteq J(R_p)$.

To see that $R_p P$ is the only maximal ideal of R_p , let X be an ideal of R_p which is not contained in $R_p P$.

Then $X \cap R \not\subseteq P$ (since $R_p(X \cap R) = X$), and if $\phi: R \rightarrow R/P$ denotes the natural surjection, $\phi(X \cap R)$ is a non-zero ideal of the prime Noetherian ring R/P , so by Goldie's theorem $\phi(X \cap R)$ contains a regular element of R/P , i.e. $(X \cap R) \cap C(P) \not\subseteq \phi$, so $X = R_p$. Thus $R_p P$ is the only maximal ideal of R_p , and $J(R_p) = R_p P$.

Now, R/P embeds in $R_p/R_p P$ by means of the map $r+P \rightarrow r+R_p P$. If $c+P$ is a regular element of R/P , i.e. if $c \in C(P)$, then $c+R_p P$ has inverse $c^{-1}+R_p P$ in $R_p/R_p P$; furthermore each element of $R_p/R_p P$ is of the form $(c+R_p P)^{-1}(a+R_p P)$ where $c \in C(P)$ and

and $a \in R$. By theorem 1.1, R_p/R_pP is isomorphic to the left quotient ring of R/P , which by theorem 1.28 of [3] is simple Artinian.

Therefore, $R_p/J(R_p)$ is simple Artinian, and R_p is local.

1.4 Definition:

Let R be a semiprime left Goldie ring, M a left R -module. Then the singular submodule $Z(M)$ is given by $Z(M) = \{m \in M \mid cm = 0 \text{ for some } c \in C(0)\}$.

The reduced rank $\rho_R(M)$ of M is defined to be the Goldie dimension of $M/Z(M)$.

Remark:

(i) The fact that $Z(M)$ is a submodule of M is a consequence of the left Ore condition on $C(0)$.

(ii) Note that it is possible for the reduced rank of a module to be infinite.

The definition of reduced rank is now extended from the semiprime case as follows.

1.5 Definition:

Let R be a ring such that the nilpotent radical N of R is nilpotent and R/N is a left Goldie ring. If $k \geq 0$ is such that $N^k = 0$, then the reduced rank of a left R -module M is defined by

$$\rho_R(M) = \sum_{i=0}^{k-1} \rho_{R/N}(N^i M / N^{i+1} M)$$

where $N^0 = R$ and the reduced ranks on the right are calculated as in definition 1.4.

The next two results provide an alternative definition of the reduced rank of a module, which is used frequently in the literature. Here, the composition length of a module M will be denoted by $L(M)$.

1.6 Lemma:

Let R be a semiprime left Goldie ring with left quotient ring Q , and let M be a left R -module. Then the kernel of the homomorphism $\phi: M \rightarrow Q \otimes_R M$ given by $\phi(m) = 1 \otimes m$ is the singular submodule, $Z(M)$.

Proof:

See Proposition 2.1, p.130 of [2].

1.7 Lemma:

Let R be a semiprime left Goldie ring with left quotient ring Q , and let M be a left R -module.

Then $\dim(M/Z(M)) = L(Q \otimes_R M)$ when either side is finite, and $\dim(M/Z(M))$ is infinite iff $L(Q \otimes_R M)$ is infinite.

Proof:

Consider the exact sequence $0 \rightarrow Z(M) \rightarrow M \rightarrow M/Z(M) \rightarrow 0$. Since, by corollary 3.32 of [19], Q is a flat right R -module, the sequence

$$0 \rightarrow Q \otimes_R Z(M) \rightarrow Q \otimes_R M \rightarrow Q \otimes_R M/Z(M) \rightarrow 0$$

is also exact. But $Q \otimes_R Z(M) = 0$, which leaves the exact sequence $0 \rightarrow Q \otimes_R M \rightarrow Q \otimes_R M/Z(M) \rightarrow 0$; therefore $Q \otimes_R M$ and $Q \otimes_R M/Z(M)$ are isomorphic as left Q -modules.

It is therefore sufficient to prove the result for the case where M is torsion-free (i.e., $Z(M) = 0$).

Let M be a non-zero torsion-free module, and suppose it contains a direct sum $M_1 \oplus \dots \oplus M_n$ of n non-zero submodules. Then, the sequence

$$0 \rightarrow M_1 \oplus \dots \oplus M_{n-1} \rightarrow M_1 \oplus \dots \oplus M_n \rightarrow M_n \rightarrow 0$$

is exact, therefore so is the sequence

$$0 \rightarrow Q \otimes_R (M_1 \oplus \dots \oplus M_{n-1}) \rightarrow Q \otimes_R (M_1 \oplus \dots \oplus M_n) \rightarrow Q \otimes_R M_n \rightarrow 0.$$

Furthermore, none of the tensor products appearing are zero, since M is torsion-free, and by lemma 1.6.

This gives rise to a chain

$$0 \neq Q \otimes_R (M_1 \oplus \dots \oplus M_{n-1}) \subsetneq Q \otimes_R (M_1 \oplus \dots \oplus M_n) \subseteq Q \otimes_R M.$$

Repeating the procedure now for the sum $M_1 \oplus \dots \oplus M_{n-1}$, and so on, yields a chain

$$0 \neq Q \otimes_R M_1 \subsetneq Q \otimes_R (M_1 \oplus M_2) \subsetneq \dots \subsetneq Q \otimes_R (M_1 \oplus \dots \oplus M_n)$$

of length n of distinct Q -submodules of $Q \otimes_R M$.

Therefore, $\dim M \leq L(Q \otimes_R M)$. (1)

Now, assume that $Q \otimes_R M$ contains a chain

$$0 \neq A_1 \subsetneq A_2 \subsetneq \dots \subsetneq A_n \subseteq Q \otimes_R M$$

of distinct non-zero Q -submodules.

Since Q is semisimple Artinian, each A_i is a direct summand of A_{i+1} (by theorem 4.3 of [19]), so $Q \otimes_R M$ contains a direct sum $B_1 \oplus \dots \oplus B_n$ of non-zero submodules.

By lemma 1.6, M may be considered as an R -submodule of $Q \otimes_R M$. Clearly, the sum $\sum_{i=1}^n B_i \cap M$ is direct. In fact, each $B_i \cap M$ is non-zero, for let $c^{-1} \otimes a$ be a non-zero element of B_i , where $a \in M$.

Then, $c(c^{-1} \otimes a) = 1 \otimes a \in B_i \cap M$ (from lemma 1.6), and $1 \otimes a$ is non-zero since M is torsion-free.

Therefore, M contains a direct sum of n non-zero submodules, and

$$\dim M \geq L(Q \otimes_R M).$$

In view of (1), the proof is complete.

The next important result of the section is the following:

1.8 Theorem (Warfield [21]):

Let R be a ring with nilpotent radical N . Then R has a left

- Artinian left quotient ring iff (i) R/N is left Goldie;
(ii) N is nilpotent;
(iii) $\rho({}_R R)$ is finite;
(iv) $C(0) = C(N)$.

Proof:

Use lemma 1.7 together with theorem 3 of [21].

§3. Primary and Artinian Rings.

Let R be a left Artinian ring with Jacobson radical $J(R)$. Then $R/J(R)$ is a semisimple Artinian ring, and by theorem 1.8 of [6], has only a finite number of minimal ideals. Each minimal ideal is generated by a unique central idempotent (by theorem 1.6 of [6]), and these will be called the semiprimitive idempotents of $R/J(R)$.

The purpose of this section is to show that if $f \in R$ is such that $\phi(f)$ is a semiprimitive idempotent of $R/J(R)$, where $\phi: R \rightarrow R/J(R)$ is the natural surjection, then fRf is a primary ring. It will also be shown that a primary ring is isomorphic to a full matrix ring over a completely primary ring. Primary and completely primary rings are defined in definition 1.9.

Both these results are well-known, and they provide a useful method for proving results about left Artinian rings: first prove the assertion for completely primary rings, then for matrix rings over completely primary rings, and finally for left Artinian rings, using the fact that fRf is primary.

This method will be used in chapter 3, as will the results about idempotents and matrix units which appear in this section.

1.9 Definition:

A left Artinian ring R will be called completely primary if $R/J(R)$ is a division ring. R will be called primary if $R/J(R)$ is simple Artinian.

Note that if R is a completely primary ring then $J(R)$ is the unique maximal one-sided ideal of R .

1.10 Theorem:

A primary ring is isomorphic to a full matrix ring over a completely primary ring.

Proof:

Let R be a primary ring. Then $R/J(R)$ is simple Artinian, so by the Wedderburn structure theorem, $R/J(R) \cong M_n(D)$ where D is a division ring. Now, by theorem 1, p.55 of [11], $R \cong M_n(B)$ where $B/J(B) \cong D$. Thus, B is a completely primary ring.

1.11 Definition:

A subset $\{e_{ij} \mid i, j = 1, \dots, n\}$ of a ring R is called a set of matrix units in R if $\sum_{i=1}^n e_{ii} = 1$ and $e_{ij} e_{kl} = e_{il} \delta_{jk}$, where δ_{jk} is the Kronecker delta.

1.12 Lemma:

Let R be a ring which possesses two sets of matrix units $\{e_{ij} \mid i, j = 1, \dots, s\}$ and $\{f_{kl} \mid k, l = 1, \dots, t\}$, such that the rings

$e_{ii}Re_{ii}$ and $f_{kk}Rf_{kk}$ are completely primary, for each $1 \leq i \leq s$, $1 \leq k \leq t$.

Then $s = t$ and there exists a unit u of R such that $f_{ij} = u^{-1}e_{ij}u$, for $i, j = 1, \dots, s$.

Proof:

See [11], theorem 3, p.59.

Recall that an idempotent element of a ring is said to be primitive if it cannot be written as the sum of two non-zero orthogonal idempotents.

1.13 Lemma:

Let R be a ring with two sets of primitive orthogonal idempotents $\{e_i \mid i = 1, \dots, s\}$ and $\{f_j \mid j = 1, \dots, t\}$ such that $\sum_{i=1}^s e_i = 1 = \sum_{j=1}^t f_j$, and the rings e_iRe_i and f_jRf_j are completely primary, for $1 \leq i \leq s$, $1 \leq j \leq t$.

Then $s = t$ and if the f_j are suitably ordered, then there exists a unit u of R such that $u^{-1}e_iu = f_i$, for all $i = 1, \dots, s$.

Proof:

See [11], theorem 2, p.59.

1.14 Lemma:

Let R be a left Artinian ring, and let e be a primitive idempotent of R . Then eRe is a completely primary ring.

Proof:

To show that eRe is left Artinian, let $(I_n)_{n \geq 0}$ be a descending sequence of left ideals of eRe . Then, $(RI_n)_{n \geq 0}$ is a descending sequence of left ideals of R , therefore $RI_k = RI_{k+1}$ for all k greater than some $m \in \mathbb{N}$. But since $I_n \subseteq eRe$ for all $n \geq 0$, $eReI_k = eReI_{k+1}$ for all $k \geq m$, i.e. $I_k = I_{k+1}$ for all $k \geq m$, and eRe is left Artinian.

Therefore, $\frac{eRe}{J(eRe)}$ is a semisimple Artinian ring. If it has a proper left ideal I , then by [6], theorem 1.12, I is generated by an idempotent \bar{e}_1 . Thus, with ϕ denoting the natural surjection, $\phi(e) = \bar{e}_1 + (\phi(e) - \bar{e}_1)$, and \bar{e}_1 and $\phi(e) - \bar{e}_1$ are orthogonal idempotents. By proposition 5, p.54 of [11], there exist orthogonal idempotents e_1 and e_2 of eRe such that $\bar{e}_1 = \phi(e_1)$, $\phi(e) - \bar{e}_1 = \phi(e_2)$ and $e_1 + e_2 = e$.

This contradicts the primitivity of e , therefore no such left ideal I exists, and eRe is completely primary.

1.15 Theorem:

Let R be a left Artinian ring with Jacobson radical $J(R)$, and let $e \in R$ be an idempotent. Then $J(eRe) = e J(R)e = eRe \cap J(R)$.

Furthermore, if e is such that $\phi(e)$ is a semiprimitive idempotent of $R/J(R)$ (where ϕ is the natural surjection) then eRe is a primary ring.

Proof:

The first assertion is standard - see proposition 1, p.48 of [11].

To prove the second assertion, denote $\phi(e)$ by \bar{e} and consider the map $\psi: eRe \rightarrow \bar{e} R/J(R) \bar{e}$ given by $\psi(ere) = \bar{e}(r+J(R))\bar{e}$.

Since $\bar{e}(r+J(R))\bar{e} = ere + J(R)$, ψ is the restriction to eRe of the natural surjection from R to $R/J(R)$, and is therefore a well-defined ring homomorphism. It is clearly surjective, and has kernel $eRe \cap J(R)$. By the first part of the theorem, $eRe \cap J(R) = J(eRe)$, therefore $\frac{eRe}{J(eRe)} \cong \bar{e} R/J(R) \bar{e}$.

Since \bar{e} is central in $R/J(R)$, $\bar{e} R/J(R) \bar{e} = R/J(R) \bar{e}$, and since \bar{e} is semiprimitive, $R/J(R) \bar{e}$ is a minimal ideal of the semisimple Artinian ring $R/J(R)$. Therefore $R/J(R) \bar{e}$ is a simple Artinian ring, and eRe is primary.

§4. Skew Laurent Polynomial Rings.

Let R be a ring, $\alpha: R \rightarrow R$ an automorphism, and x an indeterminate. Then the skew polynomial ring $R[x, \alpha]$ is defined to be the set of polynomials of the form $\sum_{i=0}^m r_i x^i$ ($m \geq 0$, $r_i \in R$) equipped with the usual addition for polynomials, and multiplication subject to the rule $xr = \alpha(r)x$, where $r \in R$.

It is easily seen that the set $\{x^i\}_{i \geq 1}$ is a left Ore subset of $R[x, \alpha]$, so as in §2, the left localization of $R[x, \alpha]$ at $\{x^i\}_{i \geq 1}$ exists; it is called the skew Laurent polynomial ring and is denoted

by $R[x, x^{-1}, \alpha]$. Since α is surjective, each element of $R[x, x^{-1}, \alpha]$ may be written in the form $\sum_{i=n}^m r_i x^i$ where $n, m \in \mathbb{Z}$.

This ring has been studied by Jordan in his paper [15], in which he finds sufficient conditions on R for $R[x, x^{-1}, \alpha]$ to be semi-primitive, primitive and Jacobson. Recall that a ring R is said to be Jacobson if each prime ideal is an intersection of primitive ideals.

He also finds, in [17], necessary and sufficient conditions for $R[x, x^{-1}, \alpha]$ to be simple.

It is the purpose of this section to present all these results: in chapter 5 they will be generalized to the case where $\alpha: R \rightarrow R$ is a monomorphism, not necessarily surjective.

1.16 Definition: ([15])

An ideal I of R is said to be an α -ideal if $\alpha(I) = I$. An α -ideal I of R is said to be α -prime if for all α -ideals A, B of R , $AB \subseteq I$ implies either $A \subseteq I$ or $B \subseteq I$. R is said to be α -prime if 0 is an α -prime ideal.

1.17 Proposition:

If R is left Noetherian and α -prime then $R[x, x^{-1}, \alpha]$ is semi-primitive.

Proof:

See proposition 2 of [15].

1.18 Definition: [15]

(i) The automorphism α is said to be stiff on R if for all non-zero ideals I of $R[x, x^{-1}, \alpha]$, $I \cap R \neq 0$.

(ii) α is said to be rigid on R if the mapping θ from the collection of ideals of $R[x, x^{-1}, \alpha]$ to the collection of α -ideals of R given by $\theta(I) = I \cap R$ is a bijection.

(iii) R is said to be α -primitive if there exists a maximal left ideal M of R which contains no non-zero α -ideals of R .

(iv) R is said to be αG if it is α -prime and the intersection of all the non-zero α -prime ideals of R is non-zero.

1.19 Theorem:

If R is α -primitive and α is stiff on R then $R[x, x^{-1}, \alpha]$ is left primitive.

Proof:

Theorem 1 of [15].

1.20 Theorem:

If R is left Noetherian, R is αG and α is stiff on R , then $R[x, x^{-1}, \alpha]$ is left primitive.

Proof:

See theorem 2 of [15].

1.21 Remark:

In his paper [15], Jordan shows that theorems 1.19 and 1.20 are

logically independent, and that $R[x, x^{-1}, \alpha]$ can be primitive without R being α -primitive, or without R being αG .

1.22 Theorem:

If R is left Noetherian and α is rigid on R then $R[x, x^{-1}, \alpha]$ is a Jacobson ring.

Proof:

See theorem 5 of [15].

1.23 Definition:

The automorphism $\alpha: R \rightarrow R$ is said to be inner if there exists a unit c of R such that, for each $r \in R$, $\alpha(r) = c^{-1}rc$. It is said to be power-inner if, for some $n \geq 0$, α^n is inner.

1.24 Theorem:

$R[x, x^{-1}, \alpha]$ is simple iff both the following hold:

- (i) R has no proper α -ideals,
- (ii) α is not power-inner.

Proof:

See theorem 1 of [17].

§5. Definition and Basic Properties of $A(k, \alpha)$.

Let $\alpha: R \rightarrow R$ be a monomorphism, not necessarily surjective. As

mentioned in the introduction, the object here is to construct an overring $A(R, \alpha)$ of R , to which α extends as an automorphism, and which is minimal among all overrings of R with that property.

The first step is the construction of the skew polynomial ring $R[x, \alpha]$, which is defined (as in section 4) to be the set of all polynomials of the form $\sum_i r_i x^i$ with the usual addition, and multiplication governed by the rule $xr = \alpha(r)x$.

As in section 4, the set $\{x^i\}_{i \geq 1}$ is a left Ore subset of $R[x, \alpha]$, so it is possible to form the skew Laurent polynomial ring $R[x, x^{-1}, \alpha]$ - however, since α is not surjective, the elements of $R[x, x^{-1}, \alpha]$ are finite sums of elements of the form $x^{-j}rx^i$ where $i, j \in \mathbb{N}$ and $r \in R$. Note that multiplication is now given by $xr = \alpha(r)x$ and $rx^{-1} = x^{-1}\alpha(r)$.

Now, $A(R, \alpha)$ is defined to be the subring $\{x^{-i}rx^i \mid r \in R, i \geq 0\}$ of $R[x, x^{-1}, \alpha]$. To see that $A(R, \alpha)$ is indeed a subring, all that is necessary is to observe that, for any $n \in \mathbb{N}$, $i \geq 0$ and $r \in R$, $x^{-i}rx^i = x^{-(i+n)}\alpha^n(r)x^{i+n}$. So, if $x^{-i}rx^i$ and $x^{-j}sx^j$ are elements of $A(R, \alpha)$, then

$$x^{-i}rx^i + x^{-j}sx^j = x^{-(i+j)}[\alpha^j(r) + \alpha^i(s)]x^{i+j} \in A(R, \alpha)$$

$$\text{and} \quad (x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{i+j} \in A(R, \alpha).$$

Thus, $A(R, \alpha)$ is a subring of $R[x, x^{-1}, \alpha]$.

The monomorphism α is then extended to $A(R, \alpha)$ by defining $\alpha(x^{-i}rx^i) = x^{-i}\alpha(r)x^i$. Since, for any $i \geq 0$ and $r \in R$,

$\alpha(x^{-(i+1)}_r x^{i+1}) = x^{-(i+1)}_{\alpha(r)} x^{i+1} = x^{-i}_r x^i$, α is actually an automorphism of $A(R, \alpha)$. No confusion should arise from the fact that α denotes both the monomorphism on R and its extension to $A(R, \alpha)$.

Now, if S is another overring of R to which α extends as an automorphism, then $A(R, \alpha)$ may be embedded in S by the map $x^{-i}_r x^i \mapsto \alpha^{-i}(r)$. Thus, $A(R, \alpha)$ is, up to isomorphism, the minimal overring of R to which α extends as an automorphism.

The next few results, all of which appear in Jordan's paper [16], summarize some of the elementary properties of $A(R, \alpha)$.

1.25 Proposition:

An element $x^{-i}_r x^i$ of $A(R, \alpha)$ is regular iff for all $n \geq 0$, $\alpha^n(r)$ is a regular element of R .

Proof:

Assume $x^{-i}_r x^i$ is regular, let $n \geq 0$, and let $s \in R$ be such that $\alpha^n(r)s = 0$. Then $(x^{-i}_r x^i)(x^{-(i+n)}_s x^{i+n}) = x^{-(i+n)}_{\alpha^n(r)} s x^{i+n} = 0$. Hence $x^{-(i+n)}_s x^{i+n} = 0$, therefore $s = 0$ and $\alpha^n(r)$ is regular.

Conversely, if $\alpha^n(r)$ is regular in R for all $n \geq 0$, and $(x^{-i}_r x^i)(x^{-j}_s x^j) = 0$ for some $j \geq 0$, $s \in R$, then $x^{-(i+j)}_{\alpha^j(r)} \alpha^j(s) x^{i+j} = 0$, which means that $\alpha^j(r) \alpha^j(s) = 0$, and therefore that $s = 0$, since $\alpha^j(r)$ is regular and α is a monomorphism. Hence, $x^{-j}_s x^j = 0$, and $x^{-i}_r x^i$ is regular.

1.26 Corollary:

$A(R, \alpha)$ is a domain iff R is a domain.

Proof:

Clear, from proposition 1.25.

1.27 Proposition:

An element $x^{-i}rx^i$ of $A(R,\alpha)$ is a unit iff for some $n \geq 0$, $\alpha^n(r)$ is a unit of R .

Proof:

If $x^{-i}rx^i$ is a unit of $A(R,\alpha)$ then, for some $s \in R$ and $j \geq 0$, $(x^{-i}rx^i)(x^{-j}sx^j) = 1$. Therefore $x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{i+j} = 1$, so $\alpha^j(r)\alpha^i(s) = 1$. Similarly, $\alpha^i(s)\alpha^j(r) = 1$, and $\alpha^j(r)$ is a unit of R .

The converse is similar.

1.28 Proposition:

If R is commutative then $A(R,\alpha)$ is commutative.

Proof:

This is a direct consequence of the definition of multiplication in $A(R,\alpha)$.

1.29 Remark:

Consider the skew Laurent polynomial ring $A(R,\alpha)[x,x^{-1},\alpha]$, where α denotes the extension to an automorphism of $A(R,\alpha)$.

Since $A(R,\alpha)$ is a subring of $R[x,x^{-1},\alpha]$, it follows that $A(R,\alpha)[x,x^{-1},\alpha] \subseteq R[x,x^{-1},\alpha]$. But since R is a subring of $A(R,\alpha)$, and α extends from R to $A(R,\alpha)$, $R[x,x^{-1},\alpha] = A(R,\alpha)[x,x^{-1},\alpha]$.

Using this observation, in chapter 5 the results about skew polynomial rings presented in the previous section will be generalized to the case where α is not necessarily surjective.

§6. Chain Conditions in $A(R, \alpha)$.

In order to examine chain conditions and ideal-theoretic properties in $A(R, \alpha)$, it is clearly necessary to determine how a left ideal of $A(R, \alpha)$ is related to the left ideal structure of R .

Let I be a left ideal of $A(R, \alpha)$. The best way to visualize I in terms of R is to define, for each $i \geq 0$, a set $I_i \subseteq R$ by putting $I_i = \{r \in R \mid x^{-i}rx^i \in I\}$. Then, I is given by the union $I = \bigcup_{i \geq 0} x^{-i}I_i x^i$, and the sequence of subsets $(I_i)_{i \geq 0}$ of R has some special properties.

Firstly, since for any $i \geq 0$ and $r \in R$, $x^{-i}rx^i = x^{-(i+1)}\alpha(r)x^{i+1}$, it is clear that $r \in I_i$ iff $\alpha(r) \in I_{i+1}$ - in other words, $\alpha^{-1}(I_{i+1}) = I_i$. This provides the motivation for the following definition:

1.30 Definition: [16]

A sequence $(I_i)_{i \geq 0}$ of subsets of R such that for all $i \geq 0$, $\alpha^{-1}(I_{i+1}) = I_i$ is called an α -sequence.

Now let $r, s \in I_i$ for some $i \geq 0$. Then $x^{-i}rx^i$ and $x^{-i}sx^i$ are elements of I , and since I is a left ideal, $x^{-i}(r-s)x^i \in I$, whence $r-s \in I_i$. Also, if $t \in R$ then, since I is a left ideal of $A(R, \alpha)$, $(x^{-i}tx^i)(x^{-i}rx^i) = x^{-i}trx^i \in I$, so $tr \in I_i$ and I_i is a left ideal of R . Furthermore, since $(I_i)_{i \geq 0}$ is an α -sequence, $\alpha^n(I_i) \subseteq I_{i+n}$

for any $n \geq 0$, and since I_{i+n} is a left ideal, $R\alpha^n(I_i) \subseteq I_{i+n}$.
 Again, because $(I_i)_{i \geq 0}$ is an α -sequence, $\alpha^{-n}(R\alpha^n(I_i)) \subseteq I_i$,
 so that $\bigcup_{n \geq 0} \alpha^{-n}(R\alpha^n(I_i)) \subseteq I_i$. Hence the following definition arises:

1.31 Definition: [16]

A left ideal I of R is said to be closed if $\bigcup_{n \geq 0} \alpha^{-n}(R\alpha^n(I)) \subseteq I$.

Thus, given a left ideal I of $A(R, \alpha)$, there exists an α -sequence $(I_i)_{i \geq 0}$ of closed left ideals of R such that $I = \bigcup_{i \geq 0} x^{-i} I_i x^i$. This correspondence is now made more precise.

1.32 Definition: [16]

Let $(I_i)_{i \geq 0}$ and $(J_i)_{i \geq 0}$ be α -sequences of closed left ideals of R . Then, define a relation " \leq " on the set of α -sequences of closed left ideals of R by putting $(I_i)_{i \geq 0} \leq (J_i)_{i \geq 0}$ iff $I_i \subseteq J_i$ for all $i \geq 0$.

It is clear that " \leq " is a relation of partial order.

1.33 Theorem:

There exists an order-preserving bijection, r , from the partially ordered set of left ideals of $A(R, \alpha)$ to the partially ordered set of α -sequences of closed left ideals of R , given by

$$r(I) = (I_i)_{i \geq 0} \quad \text{where} \quad I_i = \{r \in R \mid x^{-i} r x^i \in I\}.$$

The inverse map, Δ , is given by

$$\text{for any } \alpha \in \sum_{i \geq 0} (I_i) \alpha = \sum_{i \geq 0} x^{-i} I_i x^i,$$

Again, because $I_i \subseteq I_{i+1}$

Δ is also order-preserving.

Proof:

1.31 Definition: [16]

See theorem 4.7 of [16].

A left ideal

1.34 Proposition:

If I is a left annihilator ideal of R then I is closed.

Proof:

See lemma 4.2 of [16].

1.35 Definition:

The following result gives an important method of constructing α -sequences of closed left ideals.

1.35 Proposition:

Let I, J be closed left ideals of R . For each $k \geq 0$, let

$$\rho_k(I) = \bigcup_{n \geq 0} \alpha^{-n}(R \alpha^{n+k}(I)). \text{ Then}$$

(i) $\rho_k(I)$ is a closed left ideal for all $k \geq 0$;

1.37

(ii) $(\rho_k(I))_{k \geq 0}$ is an α -sequence;

(iii) $\rho_k(\rho_m(I)) = \rho_{k+m}(I)$ for all $k, m \geq 0$;

(iv) if $I \subseteq J$ then $\rho_k(I) \subseteq \rho_k(J)$ for all $k \geq 0$.

(i) : proposition 4.4 of [16];

(ii) : proposition 4.5 of [16];

(iii) : lemma 5.1 of [16].

The inverse map, Δ , is given by

$$\Delta((I_i)_{i \geq 0}) = \bigcup_{i \geq 0} x^{-i} I_i x^i,$$

and is also order-preserving.

Proof:

See theorem 4.7 of [16].

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If I is a left annihilator ideal of R then I is closed.

Proof:

See lemma 4.2 of [16].

The following result gives an important method of constructing α -sequences of closed left ideals.

1.35 Proposition:

Let I, J be closed left ideals of R . For each $k \geq 0$, let $\rho_k(I) = \bigcup_{n \geq 0} \alpha^{-n}(R\alpha^{n+k}(I))$. Then

- (i) $\rho_k(I)$ is a closed left ideal for all $k \geq 0$;
- (ii) $(\rho_k(I))_{k \geq 0}$ is an α -sequence;
- (iii) $\rho_k(\rho_m(I)) = \rho_{k+m}(I)$ for all $k, m \geq 0$;
- (iv) if $I \subsetneq J$ then $\rho_k(I) \subsetneq \rho_k(J)$ for all $k \geq 0$.

Proof:

- (i), (ii) : proposition 4.4 of [16];
- (iii) : proposition 4.5 of [16];
- (iv) : lemma 5.1 of [16].

1.36 Proposition:

Let $(I_i)_{i \geq 0}$ be an α -sequence of closed left ideals of R .
Then for all $i, k \geq 0$, $\rho_k(I_i) \subseteq I_{i+k}$.

Proof:

See proposition 4.6 of [16].

It is now possible to establish the first result about chain conditions in $A(R, \alpha)$.

1.37 Theorem:

If R is left Artinian then $A(R, \alpha)$ is left Artinian.

Proof:

See [16], corollary 5.3 and theorem 5.2.

Having seen that the left Artinian condition is preserved on passage from R to $A(R, \alpha)$, the next property to look at is the left Noetherian one. This is much less convenient.

1.38 Definition: [16]

An α -sequence $(I_i)_{i \geq 0}$ of closed left ideals of R is said to be stable if there exists $n \geq 0$ such that for all $i \geq n$, $\rho_1(I_i) = I_{i+1}$.

Note that the condition $\rho_1(I_i) = I_{i+1}$ for all $i \geq n$ is equivalent to the condition $\rho_i(I_n) = I_{i+n}$ for all $i \geq 0$, by proposition 1.35.

1.39 Definition:

The pair (R, α) will be called left Jordan if

1.36 Proposition:

Let $(I_i)_{i \geq 0}$ be an α -sequence of closed left ideals of R .
Then for all $i, k \geq 0$, $\rho_k(I_i) \subseteq I_{i+k}$.

Proof:

See proposition 4.6 of [16].

It is now possible to establish the first result about chain conditions in $A(R, \alpha)$.

1.37 Theorem:

If R is left Artinian then $A(R, \alpha)$ is left Noetherian.

Proof:

See [16], corollary 5.3 and theorem 5.2.

Having seen that the left Artinian condition may be transferred from R to $A(R, \alpha)$, the next question is whether $A(R, \alpha)$ is a commutative, Noetherian one. This is much less convenient.

1.38 Definition: [16] $K[y]$ to be the K -algebra endomorphism such that

$\alpha(y) = y$, and $\alpha(x) = yx$ denote the ideal generated by y .

An α -sequence $(I_i)_{i \geq 0}$ is called **stable** if there exists $n \geq 0$ such that $I_i = \alpha^n(I_0)$ for each $i \geq n$. It is easily seen that the sequence $(I_i)_{i \geq 0}$ given by $I_i = \alpha^i(I_0)$ for each $i \geq 0$ is an α -sequence of closed (left) ideals.

Note that for each $n \geq 0$, $R \alpha^{n+1}(\langle y \rangle) \subseteq \langle y^{2^{n+1}} \rangle$, and since $\alpha^n(y) = y^{2^n}$, it is clear that $y \in \alpha^{-n}(R \alpha^{n+1}(\langle y \rangle))$. Thus, the sequence $(I_i)_{i \geq 0}$

is not stable, so by theorem 1.40, $A(K[y], \alpha)$ is not Noetherian.

1.36 Proposition:

Let $(I_i)_{i \geq 0}$ be an α -sequence of closed left ideals of R .
Then for all $i, k \geq 0$, $\rho_k(I_i) \subseteq I_{i+k}$.

Proof:

See proposition 4.6 of [16].

It is now possible to establish the first result about chain conditions in $A(R, \alpha)$.

1.37 Theorem:

If R is left Artinian then $A(R, \alpha)$ is left Artinian.

Proof:

See [16], corollary 5.3 and theorem 5.2.

Having seen that the left Artinian condition is preserved on passage from R to $A(R, \alpha)$, the next property to look at is the left Noetherian one. This is much less convenient.

1.38 Definition: [16]

An α -sequence $(I_i)_{i \geq 0}$ of closed left ideals of R is said to be stable if there exists $n \geq 0$ such that for all $i \geq n$, $\rho_1(I_i) = I_{i+1}$.

Note that the condition $\rho_1(I_i) = I_{i+1}$ for all $i \geq n$ is equivalent to the condition $\rho_i(I_n) = I_{i+n}$ for all $i \geq 0$, by proposition 1.35.

1.39 Definition:

The pair (R, α) will be called left Jordan if

- (i) R has acc on closed left ideals and
- (ii) every α -sequence of closed left ideals of R is stable.

Remark:

If no confusion arises, the phrase " (R, α) is left Jordan" will often be abbreviated to " R is left Jordan".

1.40 Theorem:

The ring $A(R, \alpha)$ is left Noetherian iff R is left Jordan.

Proof:

Theorem 5.6 of [16].

The following example shows that the left Noetherian condition may easily be lost en route from R to $A(R, \alpha)$.

1.41 Example: (cf example 5.9 of [16]).

Let K be a field, and let $K[y]$ be the polynomial ring in the indeterminate y over K . Note that $K[y]$ is a commutative, Noetherian ring.

Define $\alpha: K[y] \rightarrow K[y]$ to be the K -algebra endomorphism such that $\alpha(y) = y^2$, and let $\langle y \rangle$ denote the ideal generated by y .

Then, it is easily seen that the sequence $(I_i)_{i \geq 0}$ given by $I_i = \langle y \rangle$ for each $i \geq 0$ is an α -sequence of closed (left) ideals. But for each $n \geq 0$, $R\alpha^{n+1}(\langle y \rangle) \subseteq \langle y^{2^{n+1}} \rangle$, and since $\alpha^n(y) = y^{2^n}$, it is clear that $y \notin \alpha^{-n}(R\alpha^{n+1}(\langle y \rangle))$. Thus, the sequence $(I_i)_{i \geq 0}$ is not stable, so by theorem 1.40, $A(K[y], \alpha)$ is not Noetherian.

The final result is perhaps surprising, in view of example 1.41.

1.42 Theorem:

If R is a semiprime left Goldie ring then $A(R, \alpha)$ is a semiprime left Goldie ring.

Proof:

Corollary 7.4 of [16].

CHAPTER 2

THE IDEAL STRUCTURE OF $A(R, \alpha)$.

As was shown in the work of Jordan [16], in order to study the left ideals of $A(R, \alpha)$, it is necessary to analyse the α -sequences of closed left ideals of R .

In this chapter, the α -sequences which give rise to prime, semi-prime and nilpotent ideals of $A(R, \alpha)$ are isolated, together with the α -sequence that corresponds to the nilpotent radical of $A(R, \alpha)$. These results are then used to obtain weak conditions on R which ensure that the nilpotent radical of $A(R, \alpha)$ is nilpotent.

Similar techniques are then applied to subrings, rather than left ideals, of $A(R, \alpha)$ to show that if R has the property that all nil subrings are nilpotent with bounded index of nilpotence then $A(R, \alpha)$ has the same property.

By examining the α -sequences $\Gamma(I_k)$ ($k \in \Delta$), where $\{I_k \mid k \in \Delta\}$ is a collection of left ideals of $A(R, \alpha)$ whose sum is direct, it is shown that left Goldie dimension cannot increase on passage from R to $A(R, \alpha)$. However, an example is then constructed which shows that the ascending chain condition on annihilator left ideals need not be passed from R to $A(R, \alpha)$.

First of all, a new correspondence is introduced, which is sometimes easier to apply than the α -sequence method. This is the correspondence between the collection of α -invariant left ideals of R and the collection of α -stable left ideals of $A(R, \alpha)$, and will be quite useful later on.

§1. α -Invariant and α -Stable Ideals.

2.1 Definition:

Let R be a ring, $\alpha: R \rightarrow R$ a monomorphism. Then a left, right, or two-sided ideal I of R is said to be α -invariant if $\alpha(I) \subseteq I$.

I is said to be α -stable if I is α -invariant and $\alpha^{-1}(I) \subseteq I$.

For the purpose of the following theorem, let A denote the collection of α -stable left ideals of $A(R, \alpha)$, and let B denote the collection of α -invariant left ideals of R .

Note that " α -stable left ideal of $A(R, \alpha)$ " means a left ideal which is stable under the automorphism on $A(R, \alpha)$ which extends α from R .

2.2 Theorem:

Let I be an α -stable left ideal of $A(R, \alpha)$, J an α -invariant left ideal of R .

Define $K(I) = \{r \in R \mid x^{-i}rx^i \in I \text{ for some } i \geq 0\}$

$G(J) = \{x^{-i}ax^i \mid a \in J, i \geq 0\}$.

Then (i) $K(I) \in B$, $G(J) \in A$; $G: B \rightarrow A$ and $K: A \rightarrow B$ are order-preserving maps;

(ii) $G_0 K = \text{id}_A$ and $K_0 G(J) = \bigcup_{n \geq 0} \alpha^{-n}(J)$.

Proof:

(i) First note that $K(I) = I \cap R$. Indeed, it is obvious that

$I \cap R \subseteq K(I)$. If, on the other hand, $r \in K(I)$, then $x^{-i}rx^i \in I$ for some $i \geq 0$. Since $\alpha(I) \subseteq I$, $\alpha^i(x^{-i}rx^i) \in I$, or $x^{-i}\alpha^i(r)x^i \in I$, that is, $r \in I$.

Thus $K(I) = I \cap R$, so clearly $K(I)$ is a left ideal of R , with $\alpha(K(I)) \subseteq K(I)$.

Now consider $G(J)$. Clearly, $0 \in G(J)$ and if $a \in J$, $i \geq 0$, $(x^{-i}ax^i) + (x^{-i}(-a)x^i) = 0$, so that each element of $G(J)$ has an additive inverse in $G(J)$.

Let $r \in R$, $j \geq 0$ and $b \in J$.

Then $(x^{-j}bx^j) + (x^{-i}ax^i) = x^{-(i+j)}[\alpha^i(b) + \alpha^j(a)]x^{i+j}$, and since J is α -invariant, $\alpha^i(b) \in J$ and $\alpha^j(a) \in J$, whence $G(J)$ is closed under addition.

Similarly, $(x^{-j}rx^j)(x^{-i}ax^i) = x^{-(i+j)}[\alpha^i(r)\alpha^j(a)]x^{i+j}$, and since the left ideal J is α -invariant, $\alpha^j(a) \in J$, so that $G(J)$ is a left ideal of $A(R, \alpha)$.

To see that $G(J)$ is α -stable, let $x^{-i}ax^i \in G(J)$, where $a \in J$. Then $\alpha(x^{-i}ax^i) = x^{-i}\alpha(a)x^i$, which is an element of $G(J)$ since J is α -invariant, whence $\alpha(G(J)) \subseteq G(J)$. On the other hand, it is clear (by definition of $G(J)$) that $x^{-(i+1)}ax^{i+1} \in G(J)$. But $\alpha(x^{-(i+1)}ax^{i+1}) = x^{-i}ax^i$, and therefore $\alpha^{-1}(G(J)) \subseteq G(J)$.

It is obvious that G and K are order-preserving.

(ii) Since $K(I) = I \cap R$, $G_0 K(I) = G(I \cap R)$, so that $G_0 K(I) = \{x^{-i} a x^i \mid a \in I \cap R, i \geq 0\}$. If $x^{-i} a x^i \in G_0 K(I)$ where $a \in I \cap R$, then $\alpha^{-i}(a) \in I$ since I is α -stable; thus $x^{-i} a x^i \in I$. On the other hand, if $a \in I$, say $a = x^{-i} r x^i$, then $\alpha^i(a) \in I$ since I is α -stable, whence $r \in I \cap R$, and $a \in G(I \cap R)$.

Thus, $G_0 K(I) = I$.

Now let $r \in K_0 G(J)$. Then $r = x^{-i} a x^i$ for some $a \in J$ and $i \geq 0$, i.e. $\alpha^i(r) = a$, and $r \in \bigcup_{n \geq 0} \alpha^{-n}(J)$.

If $r \in \bigcup_{n \geq 0} \alpha^{-n}(J)$, then $\alpha^i(r) \in J$ for some $i \geq 0$, and therefore $x^{-i} \alpha^i(r) x^i \in G(J)$. But $x^{-i} \alpha^i(r) x^i = r$, so $r \in G(J) \cap R = K_0 G(J)$. Thus $K_0 G(J) = \bigcup_{n \geq 0} \alpha^{-n}(J)$.

2.3 Remark:

(i) If I and J are right (or two-sided) ideals in the above theorem, then the same method of proof shows that $G(J)$ and $K(I)$ are also right (or two-sided) ideals, and that G and K have the same properties as those asserted in the theorem.

(ii) Although $K(I)$ was only shown to be an α -invariant left ideal of R , it is in fact α -stable. To see this, let $r \in R$ be such that $\alpha(r) \in K(I)$. Then $x^{-1} \alpha(r) x \in G_0 K(I) = I$, by part (ii) of the theorem. But $x^{-1} \alpha(r) x = r$, whence $r \in I \cap R = K(I)$.

The next result gives a very useful property of the α -sequences which correspond to α -stable left ideals of $A(R, \alpha)$.

2.4 Proposition:

Let I be an α -stable left ideal of $A(R, \alpha)$, and denote $\Gamma(I)$ by $(I_i)_{i \geq 0}$. Then $I_i = I_j$ for all $i, j \geq 0$.

Proof:

Without loss of generality, assume $i > j$. Let $r \in I_j$, so that $x^{-i}rx^i \in I$. Since I is α -stable, $\alpha^{i-j}(x^{-i}rx^i) \in I$, i.e. $x^{-i}\alpha^{i-j}(r)x^i \in I$, and therefore $x^{-j}rx^j \in I$. So $I_j \subseteq I_i$.

Now, if $r \in I_i$, then $x^{-j}rx^j \in I$, and since I is α -stable, $\alpha^{j-i}(I) \subseteq I$. Therefore, $x^{j-i}(x^{-j}rx^j)x^{i-j} \in I$, i.e. $x^{-i}rx^i \in I$ and $r \in I_j$. Thus $I_i = I_j$.

Remark:

In view of theorem 2.2 and the importance, already shown, of closed left ideals, it would be interesting to know whether there is any relationship between the closed left ideals and the α -invariant left ideals of a ring. The following proposition begins to answer this question.

2.5 Proposition:

Let R be a ring, and let I be an α -stable left ideal of R . Then I is closed.

Proof:

Let $n \geq 0$. Then $\alpha^{-n}(Ra^n(I)) \subseteq \alpha^{-n}(RI)$ since I is α -invariant, therefore $\alpha^{-n}(Ra^n(I)) \subseteq I$ since I is α -stable. So I is a closed left ideal.

In general, an α -invariant left ideal does not need to be α -stable in order to be closed. For example, let $R = K[y]$ where K is a field and y an indeterminate, let $\alpha: R \rightarrow R$ be the K -endomorphism of R such that $\alpha(y) = y^2$, and let I be the ideal y^2R .

Then I is α -invariant and closed, but it is not α -stable, since $y \in \alpha^{-1}(I)$.

Note that in this example, by 1.41, R is not left Jordan. In the case where R is left Jordan, the following theorem shows precisely which α -invariant left ideals are closed.

2.6 Theorem:

Let R be a left Jordan ring and let I be an α -invariant left ideal of R . Then I is a closed left ideal iff I is α -stable.

Proof:

If I is α -stable, then I is closed, by proposition 2.5.

Conversely, suppose that I is closed. Then $\alpha^{-1}(I)$ is also closed. To see this, first note that for $n = 0$, $\alpha^{-n}(R\alpha^n(\alpha^{-1}(I))) = \alpha^{-1}(I)$ since $\alpha^{-1}(I)$ is a left ideal.

Now assume that for some $k \geq 0$, $\alpha^{-k}(R\alpha^k(\alpha^{-1}(I))) \subseteq \alpha^{-1}(I)$. Then $\alpha^{-(k+1)}(R\alpha^{k+1}(\alpha^{-1}(I))) = \alpha^{-(k+1)}(R\alpha^k(\alpha(\alpha^{-1}(I))))$

$$\subseteq \alpha^{-(k+1)}(R\alpha^k(I)) \quad \text{since } \alpha(\alpha^{-1}(I)) \subseteq I$$

$$= \alpha^{-1}(\alpha^{-k}(R\alpha^k(I)))$$

$$\subseteq \alpha^{-1}(I) \quad \text{since } I \text{ is closed.}$$

By induction, $\bigcup_{n \geq 0} \alpha^{-n}(R\alpha^n(\alpha^{-1}(I))) \subseteq \alpha^{-1}(I)$ and $\alpha^{-1}(I)$ is closed. Note that the above argument did not require R to be left Jordan, nor I to be α -invariant.

Now, for each $k \geq 0$, consider the sequences $(J_{ki})_{i \geq 0}$ of left ideals of R , where

$$J_{ki} = \begin{cases} I & \text{for } i = k \\ \rho_{i-k}(I) & \text{for } i \geq k \\ \alpha^{-(k-i)}(I) & \text{for } 0 \leq i \leq k \end{cases}$$

and ρ_i is as defined in proposition 1.35.

So the sequences look like:

$$\begin{aligned} (J_{0i})_{i \geq 0} &: I & \rho_1(I) & \rho_2(I) & \rho_3(I) & \dots \\ (J_{1i})_{i \geq 0} &: \alpha^{-1}(I) & I & \rho_1(I) & \rho_2(I) & \dots \\ (J_{2i})_{i \geq 0} &: \alpha^{-2}(I) & \alpha^{-1}(I) & I & \rho_1(I) & \dots \end{aligned}$$

and so on.

By proposition 1.35, $(J_{0i})_{i \geq 0}$ is an α -sequence of closed left ideals. Furthermore, the sequence $(J_{0i})_{i \geq 0}$ is a descending one. To see this, let $r \in \rho_1(I)$ for some $i \geq 1$, so that $\alpha^n(r) \in R\alpha^{n+1}(I)$

for some $n \geq 0$. Since I is α -invariant, $\alpha^{n+1}(I) \subseteq \alpha^{n+1-1}(I)$, so that $\alpha^n(r) \in R\alpha^{n+1-1}(I)$, and $r \in \alpha^{-n}(R\alpha^{n+1-1}(I)) \subseteq \rho_{i-1}(I)$. (Note that $\rho_0(I) = I$, since I is closed.)

As was shown above, $\alpha^{-1}(I)$ is closed, and applying that argument inductively shows that $\alpha^{-k}(I)$ is closed, for all $k \geq 0$.

Since I is α -invariant, $(\alpha^{-n}(I))_{n \geq 0}$ is an increasing sequence and this, together with the fact that $(J_{0i})_{i \geq 0}$ is a decreasing sequence, shows that $(J_{ki})_{i \geq 0}$ is a decreasing sequence, for each $k \geq 0$.

Therefore, $((J_{ki})_{i \geq 0})_{k \geq 0}$ is an increasing chain of α -sequences of closed left ideals of R .

By theorem 1.33, $(\Delta(J_{ki})_{i \geq 0})_{k \geq 0}$ is an increasing chain of left ideals of $A(R, \alpha)$. Denote the left ideal $\Delta(J_{ki})_{i \geq 0}$ by J_k , and let J denote the left ideal $\bigcup_{k \geq 0} J_k$.

We now want to find the α -sequence $r(J)$.

Let $x^{-i}rx^i \in J$ for $r \in R$, $i \geq 0$, and let $k \geq 0$ be such that $x^{-i}rx^i \in J_k$. Then $r \in J_{ki}$, by definition of J_k , and by uniqueness of the α -sequence $r(J_k)$. Since each $(J_{ki})_{i \geq 0}$ is a descending sequence, it must be true that $J_{ki} \subseteq \bigcup_{n \geq 0} \alpha^{-n}(I)$.

Thus, $r(J)_i$, (i.e., the i^{th} term of the α -sequence $r(J)$) is contained in $\bigcup_{n \geq 0} \alpha^{-n}(I)$.

On the other hand, if $r \in \bigcup_{n \geq 0} \alpha^{-n}(I)$ then $r \in \alpha^{-m}(I)$ for some $m \geq 0$, so that $r \in J_{m,0}$. But $J_{m,0} = J_{m+i,i}$. Thus $x^{-i}rx^i \in J_{m+i,i} \subseteq J$, and $\bigcup_{n \geq 0} \alpha^{-n}(I) \subseteq \Gamma(J)_i$.

Therefore, $\Gamma(J)_i = \bigcup_{n \geq 0} \alpha^{-n}(I)$ for each $i \geq 0$.

But R is left Jordan, so $A(R, \alpha)$ is left Noetherian, therefore there exists $\ell \geq 0$ with $J = J_\ell$.

Since $\Gamma(J_\ell) = (J_{\ell,i})_{i \geq 0}$, equating the ℓ^{th} term of the α -sequences $\Gamma(J)$ and $\Gamma(J_\ell)$ gives

$$I = \bigcup_{n \geq 0} \alpha^{-n}(I).$$

Thus, $\alpha^{-1}(I) \subseteq I$ and I is α -stable.

2.7 Remarks:

(i) Not all closed left ideals are α -invariant, even in the left Jordan case. For example, let K be a field, $\sigma: K \rightarrow K$ a monomorphism which is not an automorphism, and let $R = K \otimes K$. Define $\alpha: R \rightarrow R$ by $\alpha(x, y) = (y, \sigma(x))$. Then R only has two proper ideals, I_1 and I_2 where $I_1 = (0, K)$ and $I_2 = (K, 0)$. Both I_1 and I_2 are closed, also $\alpha^{-1}(I_2) = I_1$, and $\alpha^{-1}(I_1) = I_2$. Thus there are only two distinct proper α -sequences of closed ideals: $I_1, I_2, I_1, I_2, \dots$ or I_2, I_1, I_2, \dots , both of which can be seen to be stable. Thus, R is Jordan, but clearly neither I_1 nor I_2 is α -invariant.

(ii) On the other hand, α -invariant ideals need not be closed. For example, consider the ring $R = \prod_{i=1}^{\infty} R_i$ where $R_1 = R_2 = \mathbb{Z}$, the ring of integers, and $R_i = \mathbb{Q}$, the rational field, for $i \geq 3$.

Define $\alpha: R \rightarrow R$ to be the monomorphism such that $\alpha(a_1, a_2, a_3, \dots) = (a_1, a_1, a_2, a_3, \dots)$, and let J denote the ideal $(2\mathbb{Z}, 2\mathbb{Z}, \mathbb{Q}, \mathbb{Q}, \mathbb{Q}, \dots)$ of R . Clearly J is α -invariant.

But $\alpha(J) = \{(a_1, a_2, a_3, \dots) \mid a_1 = a_2; a_1, a_2, a_3 \in \mathbb{Z}; a_i \in \mathbb{Q} \text{ for } i \geq 4\}$,

$$R\alpha(J) = (2\mathbb{Z}, 2\mathbb{Z}, \mathbb{Q}, \mathbb{Q}, \dots) \quad \text{and}$$

therefore $\alpha^{-1}(R\alpha(J)) = (2\mathbb{Z}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}, \dots) \not\subseteq J$.

Thus J is not closed. Note that

$$(\mathbb{Z}, 0, 0, \dots), \quad (\mathbb{Z}, \mathbb{Z}, 0, 0, \dots), \quad (\mathbb{Z}, \mathbb{Z}, \mathbb{Q}, 0, \dots)$$

etc., is an infinite ascending chain of closed ideals, so that R is neither Noetherian nor Jordan.

§2. Prime and Semiprime Ideals.

The aim of this section is to determine precisely which α -sequences of R give rise to prime, and semiprime, ideals of $A(R, \alpha)$. In order to achieve this, we define a sort of term-wise multiplication of α -sequences; once this is done, the characterization of the prime and semiprime ideals of $A(R, \alpha)$ is quite convenient.

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Define $\alpha: R \rightarrow R$ to be the monomorphism such that $\alpha(a_1, a_2, a_3, \dots) = (a_1, a_1, a_2, a_3, \dots)$, and let J denote the ideal $(2\mathbb{Z}, 2\mathbb{Z}, \mathbb{Q}, \mathbb{Q}, \mathbb{Q}, \dots)$ of R . Clearly J is α -invariant.

But $\alpha(J) = \{(a_1, a_2, a_3, \dots) \mid a_1 = a_2; a_1, a_2, a_3 \in \mathbb{Z}; a_i \in \mathbb{Q} \text{ for } i \geq 4\}$,

$$R\alpha(J) = (2\mathbb{Z}, 2\mathbb{Z}, \mathbb{Q}, \mathbb{Q}, \dots) \quad \text{and}$$

therefore $\alpha^{-1}(R\alpha(J)) = (2\mathbb{Z}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}, \dots) \not\subseteq J$.

Thus J is not closed. Note that

$$(\mathbb{Z}, 0, 0, \dots), \quad (\mathbb{Z}, \mathbb{Z}, 0, 0, \dots), \quad (\mathbb{Z}, \mathbb{Z}, \mathbb{Q}, 0, \dots)$$

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First of all, it is necessary to find those α -sequences of left ideals of R which produce two-sided ideals of $A(R, \alpha)$.

2.8 Definition:

A right ideal I of R is said to be a closed right ideal if
$$\bigcup_{n \geq 0} \alpha^{-n}(\alpha^n(I)R) \subseteq I.$$

2.9 Proposition:

Let $(I_i)_{i \geq 0}$ be an α -sequence of closed left ideals of R . Then $\Delta((I_i)_{i \geq 0})$ is an ideal of $A(R, \alpha)$ iff each I_i is a closed right ideal.

Proof:

First assume that $I = \Delta((I_i)_{i \geq 0})$ is an ideal of $A(R, \alpha)$ and let $j, k \geq 0$.

Then $(x^{-j}I_j x^j)(x^{-(j+k)}R x^{j+k}) \subseteq I$ since I is an ideal, i.e. $x^{-(j+k)}\alpha^k(I_j)R x^{j+k} \subseteq I$

or $\alpha^k(I_j)R \subseteq I_{j+k}$.

But $(I_i)_{i \geq 0}$ is an α -sequence, therefore

$$\alpha^{-k}(\alpha^k(I_j)R) \subseteq I_j, \text{ for each } k \geq 0.$$

So I_j is a closed right ideal of R , for each $j \geq 0$.

Conversely, assume that each I_i is a closed right ideal. By theorem 1.33, I is a left ideal of $A(R, \alpha)$. Let $i, j \geq 0$ and let $r, s \in R$ such that $x^{-i}rx^i \in I$ - i.e., $r \in I_i$.

$$\text{Then } (x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{i+j}.$$

Since $\alpha^j(r) \in I_{i+j}$, which is a right ideal of R , $\alpha^j(r)\alpha^i(s) \in I_{i+j}$, thus $(x^{-i}rx^i)(x^{-j}sx^j) \in I$, and I is an ideal of $A(R, \alpha)$.

The term "closed ideal" will be used to refer to an ideal of R which is closed both as a left and as a right ideal.

2.10 Definition:

Let A, B and P be left ideals of $A(R, \alpha)$ and denote the α -sequences $\Gamma(A)$, $\Gamma(B)$ and $\Gamma(P)$ by $(A_i)_{i \geq 0}$, $(B_i)_{i \geq 0}$ and $(P_i)_{i \geq 0}$ respectively.

Then the product of $\Gamma(A)$ and $\Gamma(B)$, denoted $\Gamma(A)\Gamma(B)$, is the sequence $(A_i B_i)_{i \geq 0}$ of left ideals of R .

The notation $\Gamma(A)\Gamma(B) \subseteq \Gamma(P)$ will mean that, for each $i \geq 0$, $A_i B_i \subseteq P_i$.

Let $\Gamma(P)$ be an α -sequence of closed ideals of R . Then, $\Gamma(P)$ is said to be prime if, given any two α -sequences $\Gamma(A)$ and $\Gamma(B)$ of closed left ideals, $\Gamma(A)\Gamma(B) \subseteq \Gamma(P)$ implies that either $\Gamma(A) \subseteq \Gamma(P)$ or $\Gamma(B) \subseteq \Gamma(P)$.

Remark:

It is not claimed that the product of two α -sequences is again an α -sequence.

The following result, which characterizes the prime ideals of $A(R, \alpha)$, will be needed to prove theorem 3.9, which concerns the localization of $A(R, \alpha)$ at a prime ideal.

2.11 Theorem:

An ideal P of $A(R, \alpha)$ is prime iff the α -sequence $\Gamma(P)$ of closed ideals is prime.

Proof:

First, assume that $\Gamma(P)$ is a prime α -sequence of closed ideals of R . By proposition 2.9, P is an ideal of $A(R, \alpha)$. Let A, B be left ideals of $A(R, \alpha)$ with $AB \subseteq P$, and let $r \in A_i B_i$ for some $i \geq 0$. Then $r = \sum_{k=1}^n a_k b_k$ where $x^{-i} a_k x^i \in A$ and $x^{-i} b_k x^i \in B$ for $k = 1, \dots, n$.

But $x^{-i} r x^i = \sum_{k=1}^n (x^{-i} a_k x^i)(x^{-i} b_k x^i) \in AB$, so that $r \in (AB)_i$, and consequently $A_i B_i \subseteq (AB)_i$ for each $i \geq 0$.

Now, $AB \subseteq P$, therefore $\Gamma(AB) \subseteq \Gamma(P)$, and so $A_i B_i \subseteq P_i$ - i.e. $\Gamma(A)\Gamma(B) \subseteq \Gamma(P)$. Since $\Gamma(P)$ is prime, either $\Gamma(A) \subseteq \Gamma(P)$ or $\Gamma(B) \subseteq \Gamma(P)$, and applying Δ yields that either $A \subseteq P$ or $B \subseteq P$. Thus, P is a prime ideal of $A(R, \alpha)$.

Conversely, assume that P is a prime ideal of $A(R, \alpha)$, and let $\Gamma(A)$ and $\Gamma(B)$ be two α -sequences of closed left ideals of R such that $\Gamma(A)\Gamma(B) \subseteq \Gamma(P)$. By proposition 2.9, $\Gamma(P)$ is an α -sequence of closed ideals of R .

Let $r \in (AB)_i$ for some $i \geq 0$.

Then $x^{-i}rx^i \in AB$, i.e. $x^{-i}rx^i = \sum_{k=1}^n (x^{-i_k}a_kx^{i_k})(x^{-j_k}b_kx^{j_k})$

where $x^{-i_k}a_kx^{i_k} \in A$ and $x^{-j_k}b_kx^{j_k} \in B$ for $k = 1, \dots, n$.

Let $j = \max\{i_k, j_k \mid k=1, \dots, n\}$.

Then

$$x^{-i}rx^i = x^{-j} \left[\sum_{k=1}^n \alpha^{j-i_k} (a_k)_{\alpha}^{j-j_k} \right] x^j$$

$$\text{i.e. } x^{-(i+j)} \alpha^j(r) x^{i+j} = x^{-(i+j)} \left[\sum_{k=1}^n \alpha^{i+j-i_k} (a_k)_{\alpha}^{i+j-j_k} \right] x^{i+j}$$

$$\text{and hence } \alpha^j(r) = \sum_{k=1}^n \alpha^{i+j-i_k} (a_k)_{\alpha}^{i+j-j_k} (b_k).$$

Since $a_k \in A_{i_k}$ and $(A_i)_{i \geq 0}$ is an α -sequence,

$$\alpha^{i+j-i_k} (a_k) \in A_{i_k+i+j-i_k} = A_{i+j}. \text{ Similarly, } \alpha^{i+j-j_k} (b_k) \in B_{i+j}.$$

Thus $\alpha^j(r) \in A_{i+j} B_{i+j}$, and since $\Gamma(A)\Gamma(B) \subseteq \Gamma(P)$, $\alpha^j(r) \in P_{i+j}$.

But $(P_i)_{i \geq 0}$ is an α -sequence, so that $r \in P_i$.

So, $(AB)_i \subseteq P_i$ for each $i \geq 0$, whence $\Gamma(AB) \subseteq \Gamma(P)$.

Applying Δ yields $AB \subseteq P$ and since P is prime, either $A \subseteq P$ or $B \subseteq P$.

But this means that either $\Gamma(A) \subseteq \Gamma(P)$ or $\Gamma(B) \subseteq \Gamma(P)$, and $\Gamma(P)$ is a prime α -sequence.

2.12 Corollary:

$A(R, \alpha)$ is a prime ring iff there do not exist two non-zero α -sequences of closed left ideals of R , whose product is zero.

Proof:

Put $P = 0$ in theorem 2.11.

2.13 Example:

Let R and $\alpha: R \rightarrow R$ be as in 2.7(i), that is, $R = K \oplus K$ where K is a field and $\alpha(x, y) = (y, \sigma(x))$ where $\sigma: K \rightarrow K$ is a monomorphism which is not surjective. Then the only two proper α -sequences of closed ideals are

$$(0, K), (K, 0), (0, K) \dots$$

$$\text{and } (K, 0), (0, K), (K, 0) \dots$$

Since their product is zero, corollary 2.12 shows that $A(R, \alpha)$ is not a prime ring.

2.14 Corollary: (cf 6.1 of [16])

If R is prime then $A(R, \alpha)$ is prime.

Proof:

Let $(A_i)_{i \geq 0}$ and $(B_i)_{i \geq 0}$ be two α -sequences of closed left ideals with $A_i B_i = 0$ for each $i \geq 0$.

Since R is prime, one of the sequences, say $(A_i)_{i \geq 0}$, must admit an infinite subsequence $(A_{i_k})_{k \geq 0}$ with $A_{i_k} = 0$ for all $k \geq 0$.

Let $i \geq 0$. Then, there exists $k \geq 0$ with $i_k \geq i$, and therefore $A_i = \alpha^{i-i_k} (A_{i_k}) = 0$.

So $A_i = 0$ for all $i \geq 0$, and by corollary 2.12, $A(R, \alpha)$ is a prime ring.

We now turn our attention to the semiprime ideals of $A(R, \alpha)$.

2.15 Definition:

An α -sequence $\Gamma(P)$ of closed ideals of $A(R, \alpha)$ is said to be semiprime if for any α -sequence $\Gamma(A)$ of closed left ideals of R , $\Gamma(A)^n \subseteq \Gamma(P)$ for some $n \in \mathbb{N}$ implies that $\Gamma(A) \subseteq \Gamma(P)$.

Note that the multiplication of α -sequences in the above definition is as defined in definition 2.10.

2.16 Theorem:

An ideal P of $A(R, \alpha)$ is semiprime iff the α -sequence $\Gamma(P)$ of closed ideals is semiprime.

Proof:

First, assume that $\Gamma(P)$ is a semiprime α -sequence of closed ideals, and let A be a left ideal of $A(R, \alpha)$ which satisfies $A^n \subseteq P$ for some $n \in \mathbb{N}$. By proposition 2.9, P is an ideal of $A(R, \alpha)$.

Let $i \geq 0$, and let $r \in (A_i)^n$ (where $(A_i)_{i \geq 0} = \Gamma(A)$).

Then

$$r = \sum_{k=1}^m a_{1k} a_{2k} \dots a_{nk} \quad \text{where each}$$

$a_{\ell k} \in A_i$, i.e. $x^{-i} a_{\ell k} x^i \in A$, for $\ell = 1, \dots, n$ and $k = 1, \dots, m$.

$$\begin{aligned} \text{So } x^{-i} r x^i &= x^{-i} \left[\sum_{k=1}^m a_{1k} \dots a_{nk} \right] x^i \\ &= \sum_{k=1}^m (x^{-i} a_{1k} x^i) (x^{-i} a_{2k} x^i) \dots (x^{-i} a_{nk} x^i) \in A^n. \end{aligned}$$

Hence $(A_i)^n \subseteq (A^n)_i$. Now, $A^n \subseteq P$ implies that $(A^n)_i \subseteq P_i$ for each $i \geq 0$ and therefore that $(A_i)^n \subseteq P_i$ for each $i \geq 0$. But $\Gamma(P)$ is a semiprime α -sequence, therefore $A_i \subseteq P_i$, or $\Gamma(A) \subseteq \Gamma(P)$. Applying Δ then yields that $A \subseteq P$, and P is a semiprime ideal of $A(R, \alpha)$.

Conversely, assume that P is a semiprime ideal of $A(R, \alpha)$ and let $(A_i)_{i \geq 0}$ be an α -sequence of closed left ideals of R such that $(A_i)^n \subseteq P_i$ for all $i \geq 0$, and some $n \in \mathbb{N}$.

Let $r \in (A^n)_i$ for some $i \geq 0$, i.e. $x^{-i} r x^i \in A^n$, and

$$x^{-i} r x^i = \sum_{k=1}^m (x^{-j_{1k}} a_{1k} x^{j_{1k}}) (x^{-j_{2k}} a_{2k} x^{j_{2k}}) \dots (x^{-j_{nk}} a_{nk} x^{j_{nk}}),$$

where $x^{-j_{\ell k}} a_{\ell k} x^{j_{\ell k}} \in A$ for $\ell = 1, \dots, n$ and $k = 1, \dots, m$.

Let $j = \max\{j_{\ell k} \mid \ell = 1, \dots, n; k = 1, \dots, m\}$.

$$\text{Then } x^{-i} r x^i = x^{-j} \left[\sum_{k=1}^m \alpha^{j-j_{1k}} (a_{1k}) \dots \alpha^{j-j_{nk}} (a_{nk}) \right] x^j,$$

$$\text{and } x^{-(i+j)} \alpha^j(r) x^{i+j} = x^{-(i+j)} \left[\sum_{k=1}^m \alpha^{i+j-j_{1k}} (a_{1k}) \dots \alpha^{i+j-j_{nk}} (a_{nk}) \right] x^j,$$

$$\text{so that } \alpha^j(r) = \sum_{k=1}^m \alpha^{i+j-j_{1k}} (a_{1k}) \dots \alpha^{i+j-j_{nk}} (a_{nk}).$$

$$\text{But since } a_{\ell k} \in A_{j_{\ell k}}, \alpha^{i+j-j_{\ell k}} (a_{\ell k}) \in A_{j_{\ell k} + i + j - j_{\ell k}} = A_{i+j}$$

for each $\ell = 1, \dots, n$ and $k = 1, \dots, m$.

Thus $\alpha^j(r) \in (A_{i+j})^n \subseteq P_{i+j}$. Since $(P_i)_{i \geq 0}$ is an α -sequence, this means that $r \in P_i$, and therefore that $(A^n)_i \subseteq P_i$ for all $i \geq 0$.

Now, applying Δ gives $A^n \subseteq P$, and since P is a semiprime ideal, $A \subseteq P$ so that $\Gamma(A) \subseteq \Gamma(P)$.

Thus, $\Gamma(P)$ is a semiprime α -sequence.

2.17 Corollary:

$A(R, \alpha)$ is semiprime iff there does not exist a non-zero α -sequence $(A_i)_{i \geq 0}$ of closed left ideals of R such that for some $n \in \mathbb{N}$, $(A_i)^n = 0$ for each $i \geq 0$.

Proof:

Put $P = 0$ in the theorem.

2.18 Corollary: (cf proposition 6.1 of [16])

If R is semiprime then so is $A(R, \alpha)$.

Proof:

Let $(A_i)_{i \geq 0}$ be an α -sequence of closed left ideals of R with $(A_i)^n = 0$ for some $n \in \mathbb{N}$ and all $i \geq 0$. Since R is semiprime, $A_i = 0$ for all $i \geq 0$, so by corollary 2.17, $A(R, \alpha)$ is semiprime.

2.19 Example:

Let K be a field, $\sigma: K \rightarrow K$ a monomorphism which is not an automorphism. Let R be the upper triangular matrix ring $\begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$ and define $\alpha: R \rightarrow R$ by

$$\alpha \begin{bmatrix} k_1 & k_2 \\ 0 & k_3 \end{bmatrix} = \begin{bmatrix} \sigma(k_1) & \sigma(k_2) \\ 0 & \sigma(k_3) \end{bmatrix}.$$

Let I be the ideal $\begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ of R .

Then the sequence $(A_i)_{i \geq 0}$ where $A_i = I$ for all $i \geq 0$ is a non-zero α -sequence, but $I^2 = 0$. By the argument prior to definition 1.31, I is closed as a left ideal of R .

By corollary 2.17, $A(R, \alpha)$ is not semiprime.

§3. Sums of Ideals and Goldie Dimension.

Given a collection $(B_k)_{k \in \Delta}$ of left ideals of $A(R, \alpha)$, it would be useful to know how the α -sequence of the sum $\sum_{k \in \Delta} B_k$ compared with

the α -sequences of the individual B_k 's .

That question is investigated in this section, along with a closely related one; namely, when is such a sum direct?

The result about directness is then applied to show that if R has finite left Goldie dimension, then the left Goldie dimension of $A(R, \alpha)$ cannot exceed that of R . In the light of this result, it would be interesting to know whether the ascending chain condition for annihilator left ideals is preserved on passage from R to $A(R, \alpha)$, and example 2.25 shows that this is, in general, not the case.

The work concerning the α -sequence of a sum of ideals will be used in the next section to study the nilpotent radical of $A(R, \alpha)$.

2.20 Proposition:

Let $(B_k)_{k \in \Delta}$ be a collection of left ideals of $A(R, \alpha)$, and denote the α -sequences $\Gamma(B_k)$ and $\Gamma(\sum_{k \in \Delta} B_k)$ by $((B_k)_i)_{i \geq 0}$ and $(A_i)_{i \geq 0}$ respectively.

Then, for each $i \geq 0$, $A_i = \bigcup_{n \geq 0} \alpha^{-n}(\sum_{k \in \Delta} (B_k)_{i+n})$.

Proof:

First let $r \in \bigcup_{n \geq 0} \alpha^{-n}(\sum_{k \in \Delta} (B_k)_{i+n})$ for some $i \geq 0$.

So, for some $n \geq 0$, $\alpha^n(r) = \sum_{k \in \Delta} b_k$ where $x^{-(i+n)} b_k x^{i+n} \in B_k$

for each $k \in \Delta$, (and only a finite number of the b_k 's are non-zero).

Therefore, $x^{-(i+n)} \alpha^n(r) x^{i+n} = \sum_{k \in \Delta} x^{-(i+n)} b_k x^{(i+n)} \in \sum_{k \in \Delta} B_k$.

But $x^{-(i+n)} \alpha^n(r) x^{i+n} = x^{-i} r x^i$, so $r \in A_i$, and

$$\bigcup_{n \geq 0} \alpha^{-n} \left(\sum_{k \in \Delta} (B_k)_{i+n} \right) \subseteq A_i \quad \text{for each } i \geq 0.$$

Now let $r \in A_i$ - i.e. $x^{-i} r x^i \in \sum_{k \in \Delta} B_k$ and $x^{-i} r x^i = \sum_{k \in \Delta} x^{-j_k} b_k x^{j_k}$ where $x^{-j_k} b_k x^{j_k} \in B_k$ for each $k \in \Delta$.

Let $j = i + \max\{j_k \mid k \in \Delta \text{ and } b_k \neq 0\}$.

$$\text{Then } x^{-j} \alpha^{j-i}(r) x^j = x^{-j} \sum_{k \in \Delta} \alpha^{j-j_k}(b_k) x^j$$

$$\text{i.e. } \alpha^{j-i}(r) = \sum_{k \in \Delta} \alpha^{j-j_k}(b_k).$$

But since $b_k \in (B_k)_{j_k}$, $\alpha^{j-j_k}(b_k) \in (B_k)_j$ and therefore

$$\alpha^{j-i}(r) \in \sum_{k \in \Delta} (B_k)_j.$$

Putting $n = j-i$, this gives $\alpha^n(r) \in \sum_{k \in \Delta} (B_k)_{i+n}$ and so

$$A_i = \bigcup_{n \geq 0} \alpha^{-n} \left(\sum_{k \in \Delta} (B_k)_{i+n} \right).$$

Remark:

$(\alpha^{-n} (\sum_{k \in \Delta} (B_k)_{i+n}))_{n \geq 0}$ is in fact an ascending sequence of left ideals of R , for each $i \geq 0$.

Indeed, for any $n \geq 0$, let $\alpha^n(r) \in \sum_{k \in \Delta} (B_k)_{i+n}$.

Then $\alpha^n(r) = \sum_{k \in \Delta} b_k$ where $x^{-(i+n)} b_k x^{i+n} \in B_k$ (again, only finitely many of the b_k 's are non-zero).

Therefore, $\alpha^{n+1}(r) = \sum_{k \in \Delta} \alpha(b_k)$ and

$$x^{-(i+n+1)} \alpha(b_k) x^{i+n+1} = x^{-(i+n)} b_k x^{i+n} \in B_k,$$

hence $\alpha(b_k) \in (B_k)_{i+n+1}$, or $r \in \alpha^{-(n+1)}(\sum_{k \in \Delta} (B_k)_{i+n+1})$.

The next result shows when a sum of left ideals of $A(R, \alpha)$ is direct.

2.21 Proposition:

Let $(B_k)_{k \in \Delta}$ be a collection of left ideals of $A(R, \alpha)$, and denote the α -sequence $\Gamma(B_k)$ by $((B_k)_i)_{i \geq 0}$, for each $k \in \Delta$.

Then, the sum $\sum_{k \in \Delta} B_k$ is direct iff for each $\ell \geq 0$, the sum $\sum_{k \in \Delta} (B_k)_\ell$ is a direct sum of left ideals of R .

Proof:

Assume there exists $\ell \geq 0$ with $\sum_{k \in \Delta} (B_k)_\ell$ not a direct sum of left ideals of R .

Then there exists a finite subset $\{1, \dots, n\}$ of Δ and $0 \neq r_k \in (B_k)_\ell$

for $k = 1, \dots, n$ such that

$$\sum_{k=1}^n r_k = 0 \quad ; \quad \text{i.e.} \quad \sum_{k=1}^n x^{-1} r_k x^1 = 0 .$$

But $x^{-1} r_k x^1 \in B_k$ for each $k = 1, \dots, n$ so the sum $\sum_{k \in \Delta} B_k$ is not direct.

Now assume that $\sum_{k \in \Delta} B_k$ is not direct. Then there exist non-zero elements $x^{-j_k} r_k x^{j_k}$ of B_k , for $k = 1, \dots, n$ say, such that

$$\sum_{k=1}^n x^{-j_k} r_k x^{j_k} = 0 .$$

Let $j = \max\{j_k \mid k = 1, \dots, n\}$.

$$\text{Then } \sum_{k=1}^n x^{-j} x^{j-j_k} (r_k) x^j = 0 , \quad \text{i.e.} \quad \sum_{k=1}^n \alpha^{j-j_k} (r_k) = 0 .$$

But since $r_k \in (B_k)_{j_k}$, $\alpha^{j-j_k} (r_k) \in (B_k)_j$, and since α is a monomorphism, $\alpha^{j-j_k} (r_k) \neq 0$ for each $k = 1, \dots, n$.

Thus, $\sum_{k \in \Delta} (B_k)_j$ is not direct.

2.22 Corollary:

$A(R, \alpha)$ has finite left Goldie dimension iff there does not exist an infinite collection $((B_k)_\ell)_{\ell \geq 0}$ of non-zero α -sequences of closed

left ideals of R such that $\sum_{k \in \Delta} (B_k)_\lambda$ is direct, for each $\lambda \geq 0$.

Proof:

Immediate from proposition 2.21.

2.23 Corollary:

If R has left Goldie dimension $n < \infty$ then $A(R, \alpha)$ has left Goldie dimension at most n .

Proof:

If possible, let $(B_k)_{1 \leq k \leq n+1}$ be a collection of non-zero left ideals of $A(R, \alpha)$ whose sum is direct.

Each $((B_k)_\lambda)_{\lambda \geq 0}$ ($\equiv \Gamma(B_k)$) is a non-zero α -sequence, so for each $1 \leq k \leq n+1$, there exists $\lambda_k \geq 0$ such that if $\lambda \geq \lambda_k$ then $(B_k)_\lambda \neq 0$. This follows because if no such λ_k exists, then $((B_k)_\lambda)_{\lambda \geq 0}$ has an infinite subsequence $((B_k)_{\lambda_i})_{i \geq 0}$ with $(B_k)_{\lambda_i} = 0$ for all $i \geq 0$, and as in the proof of corollary 2.14, this implies $(B_k)_\lambda = 0$ for all $\lambda \geq 0$.

Let $\lambda = \max\{\lambda_k \mid 1 \leq k \leq n+1\}$.

Then $(B_k)_\lambda \neq 0$ for each $1 \leq k \leq n+1$ and by proposition 2.21,

$\sum_{k=1}^{n+1} (B_k)_\lambda$ is direct. This is impossible since R has left Goldie

dimension n .

2.24 Example:

Let S be a ring and let R be the product $\prod_{i=1}^{\infty} S_i$ where

$$S_i = S \text{ for all } i \geq 1.$$

Define $\alpha: R \rightarrow R$ by $\alpha(s_1, s_2, s_3, \dots) = (s_1, s_1, s_2, s_3, \dots)$, and for $j \geq 1$, let I_j be the ideal of R which has S in the j^{th} co-ordinate and zero elsewhere - e.g. $I_2 = (0, S, 0, 0, \dots)$.

Consider the α -sequence $((B_k)_\lambda)_{\lambda \geq 0}$ where $(B_k)_\lambda = I_{k+\lambda+1}$, so that the α -sequence $((B_2)_\lambda)_{\lambda \geq 0}$ looks like

$$(0, 0, S, 0, 0, \dots), \quad (0, 0, 0, S, 0, 0, \dots), \quad (0, 0, 0, 0, S, 0, \dots) \text{ etc.}$$

Clearly each I_j is closed, so that, for all $\lambda \geq 0$, $\sum_{k=1}^{\infty} (B_k)_\lambda$ is a direct sum of non-zero closed ideals of R .

By corollary 2.22, $A(R, \alpha)$ has infinite left Goldie dimension.

In view of corollary 2.23, attention is now turned to the other Goldie criterion: the ascending chain condition for annihilator left ideals. The following example shows that it is possible that R has acc on left annihilators, but that $A(R, \alpha)$ does not. The ring R concerned was used by Kerr [18] as an example of a ring with acc on annihilators, but with no bound on the lengths of chains of annihilators.

2.25 Example:

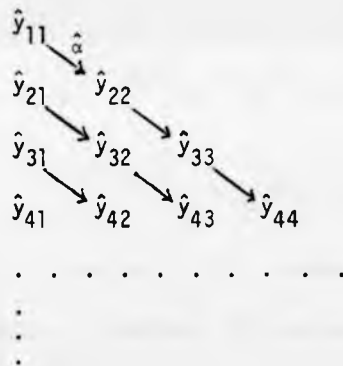
Let K be a field and let

$$\hat{Y} = \{\hat{y}_{ij} \mid i, j \in \mathbb{N}, j \leq i\}$$

be a collection of commuting indeterminates.

Let $\hat{\alpha} : K[\hat{Y}] \rightarrow K[\hat{Y}]$ be the K -monomorphism such that $\hat{\alpha}(\hat{y}_{ij}) = \hat{y}_{i+1, j+1}$.

The action of $\hat{\alpha}$ can be represented by the array:



Now consider the ideal I of $K[\hat{Y}]$ generated by $\{\hat{y}_{ij}^3, \hat{y}_{ij}\hat{y}_{ik} \mid i, j, k \in \mathbb{N}, k \neq j\}$.

Clearly, $\hat{\alpha}(I) \subseteq I$.

Also, any element of $\hat{\alpha}(K[\hat{Y}])$ is a sum of terms of the form

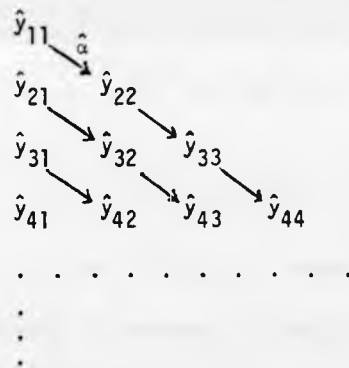
$$k \hat{y}_{1j_1}^{n(1)} \hat{y}_{2j_2}^{n(2)} \dots \hat{y}_{pj_p}^{n(p)} \quad \text{where } k \in K \text{ and } i_\ell, j_\ell \geq 2 \text{ for } \ell = 1, \dots, p.$$

$$\hat{Y} = \{\hat{y}_{ij} \mid i, j \in \mathbb{N}, j \leq i\}$$

be a collection of commuting indeterminates.

Let $\hat{\alpha} : K[\hat{Y}] \rightarrow K[\hat{Y}]$ be the K -monomorphism such that $\hat{\alpha}(\hat{y}_{ij}) = \hat{y}_{i+1, j+1}$.

The action of $\hat{\alpha}$ can be represented by the array:



Now consider the ideal I of $K[\hat{Y}]$ generated by $\{\hat{y}_{ij}^3, \hat{y}_{ij}\hat{y}_{ik} \mid i, j, k \in \mathbb{N}, k \neq j\}$.

Clearly, $\hat{\alpha}(I) \subseteq I$.

Also, any element of $\hat{\alpha}(K[\hat{Y}])$ is a sum of terms of the form

$$k \hat{y}_{i_1 j_1}^{n(1)} \hat{y}_{i_2 j_2}^{n(2)} \dots \hat{y}_{i_p j_p}^{n(p)} \quad \text{where } k \in K \text{ and } i_\ell, j_\ell \geq 2 \text{ for } \ell = 1, \dots, p.$$

So if $f \in K[\hat{Y}]$ and $\hat{\alpha}(f) \in I$, then

$$\hat{\alpha}(f) = \sum_m g_m \hat{y}_{a_m} \hat{y}_{b_m} \hat{y}_{c_m} \hat{y}_{d_m} \hat{y}_{e_m} \hat{y}_{f_m} + \sum_l h_l \hat{y}_{i_l} \hat{y}_{j_l} \hat{y}_{k_l}$$

where $g_m, h_l \in \hat{\alpha}(K[\hat{Y}])$ for each m, l , and none of the $a_m, b_m, c_m, d_m, e_m, f_m, i_l, j_l, k_l$ is equal to 1.

Thus,

$$f = \sum_m \hat{\alpha}^{-1}(g_m) \hat{y}_{a_m-1} \hat{y}_{b_m-1} \hat{y}_{c_m-1} \hat{y}_{d_m-1} \hat{y}_{e_m-1} \hat{y}_{f_m-1} \\ + \sum_l \hat{\alpha}^{-1}(h_l) \hat{y}_{i_l-1} \hat{y}_{j_l-1} \hat{y}_{k_l-1}$$

So $f \in I$ and $\hat{\alpha}^{-1}(I) \subseteq I$ - whence I is $\hat{\alpha}$ -stable.

Therefore, $\hat{\alpha}$ defines, in a natural way, a monomorphism

$$\alpha : \frac{K[\hat{Y}]}{I} \rightarrow \frac{K[\hat{Y}]}{I}.$$

$\frac{K[\hat{Y}]}{I}$ will be denoted by R , Y will denote the image of \hat{Y} in R , and y_{ij} will denote the image of \hat{y}_{ij} in R .

The relationships thus created between the (y_{ij}) may be summed up by saying that, in the above array, the product of any two distinct terms in the same row is zero, as is the product of any three terms in the array.

The following argument, due to Kerr [18], shows that the ring R

has acc on annihilator ideals. It achieves this by explicitly determining the form of all the annihilator ideals of R .

First, it is necessary to establish some notation. R can be given a graded structure by making all the elements of Y homogeneous of degree 1, and all the elements of K homogeneous of degree 0. So $R = R_0 \oplus R_1 \oplus R_2$ where R_i consists of all the homogeneous elements of degree i . The set Y forms a basis over K for R_1 , and the set $\{y_{ij}y_{kl} \mid \text{either } i \neq k, \text{ or } i = k \text{ and } k \neq l\}$ forms a basis over K for R_2 .

Let Y_n denote the set $\{y_{nj} \mid j \leq n\}$, for each $n \in \mathbb{N}$.

The precise form of all the annihilator ideals is summarized by the following:

Summary:

There are three separate cases to be considered. Let $S \subset R$, and assume that $S \neq \{0\}$.

(i) If $S \not\subset R_1R$ then $\ell(S) = 0$;

(ii) If $S \subset R_1R$ and $S \subset R_2$ then $\ell(S) = R_1R$;

(iii) If $S \subset R_1R$ but $S \not\subset R_2$ then either $\ell(S) = R_2$

or there exists $n \in \mathbb{N}$ and a subset T of Y_n such that $\ell(S) = TR + R_2$.

Proof:

(i) Note that $(R_1R)^3 = 0$, and that if $f \notin R_1R$, then f

has a non-zero constant term and is therefore regular. Thus, if $S \notin R_1 R$ then $\ell(S) = 0$.

(ii) Clearly $R_1 R \subset \ell(R_2) \subset \ell(S)$. Also, since any element lying outside $R_1 R$ is regular, $\ell(S) = R_1 R$.

(iii) Here, it is necessary to find $\ell(f)$ where $f \in R_1 R$ but $f \notin R_2$. Let $g \in R$ be such that $fg = 0$, and write f and g as sums of their homogeneous components:

$$f = f_0 + f_1 + f_2 \quad \text{and} \quad g = g_0 + g_1 + g_2.$$

Since $f \in R_1 R$, $f_0 = 0$ and since g is not regular, $g_0 = 0$. So $fg = (f_1 + f_2)(g_1 + g_2) = f_1 g_1$. Let $f_1 = \sum_{i,j} a_{ij} y_{ij}$ and let

$g_1 = \sum_{i,j} b_{ij} y_{ij}$, so that

$$\begin{aligned} fg = \sum_{i,j} a_{ij} b_{ij} y_{ij}^2 &+ \sum_{\substack{i,j,k \\ j < k}} (a_{ij} b_{ik} + a_{ik} b_{ij}) y_{ij} y_{ik} \\ &+ \sum_{\substack{i,j,k,l \\ i < k}} (a_{ij} b_{kl} + b_{ij} a_{kl}) y_{ij} y_{kl}. \end{aligned}$$

The second term is zero because $y_{ij} y_{ik} = 0$ for $j \neq k$, and the other two terms give:

$$a_{ij} b_{ij} = 0 \quad \text{for all } i, j \quad (1)$$

$$\text{and} \quad a_{ij} b_{kl} + a_{kl} b_{ij} = 0 \quad \text{for all } i, j, k, l \text{ with } i \neq k. \quad (2)$$

Since $f \notin R_2$, at least one of the a_{ij} (say a_{nm}), must be non-zero.

From (1), $b_{nm} = 0$ and from (2),

$$a_{ij}b_{nm} + a_{nm}b_{ij} = 0 \text{ for all } i, j \text{ with } i \neq n.$$

Thus, $b_{ij} = 0$ for all i, j with $i \neq n$; in other words, $g \in T_{nm}R + R_2$ where $T_{nm} = Y_n \setminus \{y_{nm}\}$.

Now, if $\{a_{n_p m_p}\}_{1 \leq p \leq q}$ is the set of all the non-zero a_{ij} 's, then applying the above to each $a_{n_p m_p}$ gives

$$\begin{aligned} g \in \bigcap_{p=1}^q (T_{n_p m_p} R + R_2) &= \left(\bigcap_{p=1}^q T_{n_p m_p} \right) R + R_2 \\ &= T_f R + R_2 \quad \text{where} \end{aligned}$$

T_f denotes the set $\bigcap_{p=1}^q T_{n_p m_p}$.

Thus, if $S \subset R_1 R$ but $S \not\subset R_2$, then

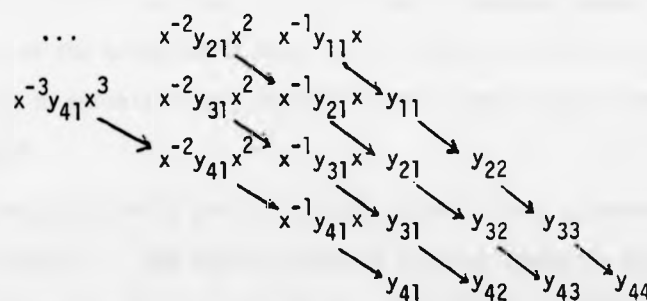
$$\ell(S) = \bigcap_{f \in S} \ell(f) = \bigcap_{f \in S} (T_f R + R_2) = \left(\bigcap_{f \in S} T_f \right) R + R_2,$$

where $\bigcap_{f \in S} T_f \subset Y_n$ for some $n \in \mathbb{N}$ - if $\bigcap_{f \in S} T_f = \emptyset$ then $\ell(S) = R_2$.

This completes the proof of the assertions appearing in the above summary.

It is now easy to see that R has acc on annihilators. Let J be a non-trivial annihilator ideal other than $R_1 R$. By the above results, $J = TR + R_2$ where $T \subset Y_n$ for some $n \in \mathbb{N}$. But T is a finite set, so J can only contain a finite number of distinct annihilator ideals. This shows that R has descending chain condition on annihilator ideals, and since R is commutative, R also has acc on annihilator ideals.

Now consider the ring $A(R, \alpha)$. The process of forming $A(R, \alpha)$ may be thought of as extending the previous array as follows:



Consider, for $m \geq 0$, an element of the form $x^{-m}y_{m+1,1}x^m$. Then, $(x^{-m}y_{m+1,1}x^m)^2 \neq 0$, but for $n \geq 0$ with $n \neq m$

$$\begin{aligned} (x^{-n}y_{n+1,1}x^n)(x^{-m}y_{m+1,1}x^m) &= x^{-(m+n)}\alpha^m(y_{n+1,1})\alpha^n(y_{m+1,1})x^{m+n} \\ &= x^{-(m+n)}y_{n+m+1,1+m}y_{n+m+1,1+n}x^{m+n} \\ &= 0. \end{aligned} \quad (3)$$

Notice that the elements appearing here are precisely those occupying the same row as y_{11} in the array.

Now let $B_n = \{x^{-m}y_{m+1,1}x^m \mid m \geq n\}$, for each $n \geq 0$. Since $B_n \supseteq B_{n+1}$ for all $n \geq 0$, certainly $\ell(B_n) \subseteq \ell(B_{n+1})$. But from (3), $x^{-n}y_{n+1,1}x^n \in \ell(B_{n+1})$ but $x^{-n}y_{n+1,1}x^n \notin \ell(B_n)$.

Thus, $(\ell(B_n))_{n \geq 0}$ is an infinite ascending sequence of annihilators of $A(R, \alpha)$.

§4. The Nilpotent Radical.

The aim of this section is to determine in complete generality the α -sequence of the nilpotent radical N of $A(R, \alpha)$, and to use this information to establish weak conditions on R which ensure that N is nilpotent.

In view of the fact that N is the sum of all the nilpotent left ideals of $A(R, \alpha)$, the approach used is first to determine the α -sequences which correspond to nilpotent left ideals of $A(R, \alpha)$, and then to use proposition 2.20 to find the α -sequence which corresponds to N . This α -sequence is itself shown to give rise to a nilpotent left ideal of $A(R, \alpha)$ in the case where every nil left ideal of R is nilpotent.

2.26 Definition:

An α -sequence $\Gamma(A)$ of closed left ideals of R is said to be nilpotent if there exists $n \in \mathbb{N}$ with $\Gamma(A)^n = 0$.

2.27 Lemma:

A left ideal A of $A(R, \alpha)$ is nilpotent iff the α -sequence $\Gamma(A)$ is nilpotent.

Proof:

Denote the α -sequence $\Gamma(A)$ by $(A_i)_{i \geq 0}$ and assume it to be nilpotent - i.e. $\Gamma(A)^n = 0$ for some $n \in \mathbb{N}$.

Let $x^{-i_k} a_k x^{i_k} \in A$ for $1 \leq k \leq n$.

Then $\prod_{k=1}^n x^{-i_k} a_k x^{i_k} = x^{-i} \prod_{k=1}^n \alpha^{-i_k}(a_k) x^i$ where

$i = \max\{i_k \mid 1 \leq k \leq n\}$. But $x^{-i} \prod_{k=1}^n \alpha^{-i_k}(a_k) x^i = x^{-i} a_k x^{i_k} \in A$, so

$\alpha^{-i_k}(a_k) \in A_i$ for each $1 \leq k \leq n$. Since $A_i^n = 0$,

$\prod_{k=1}^n \alpha^{-i_k}(a_k) = 0$. Thus $A^n = 0$ and A is a nilpotent left ideal

of $A(R, \alpha)$.

Conversely, assume that A is a nilpotent left ideal of $A(R, \alpha)$ and let $n \in \mathbb{N}$ be such that $A^n = 0$.

Let $r = a_1 a_2 \dots a_n$ where $x^{-i_k} a_k x^{i_k} \in A$ for some $i_k \geq 0$, and each $1 \leq k \leq n$.

Then $x^{-i} r x^i = \prod_{k=1}^n x^{-i_k} a_k x^{i_k} \in A^n = 0$, and therefore $r = 0$.

Thus $(A_i)^n = 0$ and $\Gamma(A)$ is a nilpotent α -sequence.

2.28 Theorem:

Let N denote the nilpotent radical of $A(R, \alpha)$ and let $(N_i)_{i \geq 0}$ denote the α -sequence $\Gamma(N)$.

Then, for all $i \geq 0$, $N_i = \bigcup_{n \geq 0} \alpha^{-n}(\sum_{k \in \Delta} (B_k)_n)$ where

$\{(B_k)_i\}_{i \geq 0} \mid k \in \Delta\}$ is the collection of all nilpotent α -sequences of closed left ideals of R .

Proof:

$N = \sum_{k \in \Delta} A_k$ where $\{A_k \mid k \in \Delta\}$ is the collection of all nilpotent left ideals of $A(R, \alpha)$.

By lemma 2.27, there is a one-to-one correspondence between the collection of nilpotent left ideals of $A(R, \alpha)$ and the collection of nilpotent α -sequences of closed left ideals of R .

By proposition 2.20, then,

$$N_i = \bigcup_{n \geq 0} \alpha^{-n}(\sum_{k \in \Delta} (B_k)_{i+n}) \quad \text{for each } i \geq 0.$$

But $\alpha(N)$ and $\alpha^{-1}(N)$ are both sums of nilpotent left ideals of $A(R, \alpha)$, so that $\alpha(N) \subseteq N$ and $\alpha^{-1}(N) \subseteq N$; thus N is α -stable.

By proposition 2.4 then, $N_i = N_j$ for all $i, j \geq 0$, and

$$N_i = \bigcup_{n \geq 0} \alpha^{-n}(\sum_{k \in \Delta} (B_k)_n) \quad \text{for each } i \geq 0.$$

2.29 Example:

Let R , $\alpha: R \rightarrow R$ and I be as in example 2.19, so that

$$R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix} \quad \text{where } K \text{ is a field, } \alpha \begin{bmatrix} k_1 & k_2 \\ 0 & k_3 \end{bmatrix} = \begin{bmatrix} \sigma(k_1) & \sigma(k_2) \\ 0 & \sigma(k_3) \end{bmatrix}$$

where $\sigma: K \rightarrow K$ is a non-surjective monomorphism, and I is the ideal

$\begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$. Then, I is a closed left ideal, $\alpha^{-1}(I) = I$ and I is the only nilpotent left ideal of R . Thus, the only nilpotent α -sequence is $(I_i)_{i \geq 0}$ where $I_i = I$ for each $i \geq 0$.

By theorem 2.28, $N = \Delta(N_i)_{i \geq 0}$ where $N_i = \bigcup_{n \geq 0} \alpha^{-n}(I) = I$.

Thus, $N = \bigcup_{i \geq 0} x^{-i} I x^i = G(I)$.

2.30 Theorem:

If R is a ring such that each nil left ideal of R is nilpotent, then $N(A(R, \alpha))$ is nilpotent.

Proof:

Denoting $N(A(R, \alpha))$ by N , theorem 2.28 gives

$$N_i = \bigcup_{n \geq 0} \alpha^{-n} \left(\sum_{k \in \Delta} (B_k)_n \right) \quad \text{where}$$

$\{(B_k)_i\}_{i \geq 0} \mid k \in \Delta\}$ is the collection of nilpotent α -sequences of closed left ideals of R .

Since each $(B_k)_n$ is a nilpotent left ideal, $\sum_{k \in \Delta} (B_k)_n$ is a nil left ideal, for each $n \geq 0$.

Now let $r \in \alpha^{-n} \left(\sum_{k \in \Delta} (B_k)_n \right)$. Then $\alpha^n(r)^m = 0$ for some $m \in \mathbb{N}$, i.e. $\alpha^n(r^m) = 0$, and $r^m = 0$ since α^n is a monomorphism. Thus $\bigcup_{n \geq 0} \alpha^{-n} \left(\sum_{k \in \Delta} (B_k)_n \right)$ is a nil left ideal of R , therefore it is nilpotent.

Thus, $(N_i)_{i \geq 0}$ is a nilpotent α -sequence, so by lemma 2.27, N is nilpotent.

2.31 Corollary:

Let R be a ring which satisfies any of the following:

- (i) R is a left Goldie ring ;
- (ii) R has acc and dcc on annihilator left ideals ;
- (iii) R has Krull dimension.

Then $N(A(R,\alpha))$ is nilpotent.

Proof:

(i) By Lanski's theorem ([3], theorem 1.35), nil subrings of a left Goldie ring are nilpotent, so the result follows by theorem 2.30.

(ii) By the theorem of Herstein and Small ([3], theorem 1.34), nil subrings of a ring with acc and dcc on left annihilators are nilpotent, and theorem 2.30 finishes the proof.

(iii) By theorem 5.1 of [10], nil subrings of a ring with Krull dimension are nilpotent, so theorem 2.30 yields the result.

Remark:

Note that conditions (ii) and (iii) do not themselves persist on passage from R to $A(R,\alpha)$. Example 2.25 shows that it is possible for R to have acc and dcc on left annihilators, and for this condition to fail in $A(R,\alpha)$, while example 3.15, which originally appeared in [16], shows that Krull dimension can also be lost.

It is not known whether R being a left Goldie ring ensures that $A(R,\alpha)$ is left Goldie.

§5. Nil Subrings.

In the previous section, it was shown that certain weak chain conditions on R are sufficient to make the nilpotent radical N of $A(R, \alpha)$ nilpotent. The object here is to use a similar approach to obtain conditions on R which would ensure that nil subrings of $A(R, \alpha)$ are nilpotent. For instance, it turns out that this is the case if R is a left Goldie ring.

To establish the main result, it is first necessary to find a way of identifying, in R , the subrings of $A(R, \alpha)$, and then to be able to determine which ones are nil and which ones are nilpotent.

Recall that if S is a nilpotent subring of R , then the index of nilpotence of S is the smallest integer k for which $S^k = 0$.

2.33 Lemma:

There is an order-preserving bijection, with an order-preserving inverse, from the partially ordered set of subrings of $A(R, \alpha)$ to the partially ordered set of α -sequences of subrings of R .

Proof:

First note that the collection S of subrings of $A(R, \alpha)$ is partially ordered by inclusion, and the collection A of α -sequences of subrings of R is partially ordered by the relation \leq , given by $(A_i)_{i \geq 0} \leq (B_i)_{i \geq 0}$ iff $A_i \subseteq B_i$ for each $i \geq 0$.

Let $S \in S$ and for each $i \geq 0$, put $S_i = \{r \in R \mid x^{-i}rx^i \in S\}$.

Then $(S_i)_{i \geq 0}$ is an α -sequence of subrings of R . Indeed, $r \in S_i$ by definition means $x^{-i}rx^i \in S$, and $x^{-i}rx^i = x^{-(i+1)}\alpha(r)x^{i+1}$, so that $r \in S_i$ iff $\alpha(r) \in S_{i+1}$. So $(S_i)_{i \geq 0}$ is an α -sequence. Now let $r, s \in S_i$. Then, $x^{-i}rx^i, x^{-i}sx^i \in S$, therefore $x^{-i}(r-s)x^i$ and $x^{-i}rsx^i \in S$, since S is a subring of $A(R, \alpha)$. Thus $r-s \in S_i$ and $rs \in S_i$, whence S_i is a subring of R for each $i \geq 0$, and $(S_i)_{i \geq 0} \in A$. If this α -sequence is denoted by $\Gamma(S)$ then clearly $\Gamma: S \rightarrow A$ is order-preserving.

Now let $(S_i)_{i \geq 0} \in A$ and let S denote the set $\bigcup_{i \geq 0} x^{-i}S_i x^i$. Then S is a subring of $A(R, \alpha)$. Indeed, if $x^{-i}rx^i, x^{-j}sx^j \in S$, then

$$x^{-i}rx^i - x^{-j}sx^j = x^{-(i+j)}(\alpha^j(r) - \alpha^i(s))x^{i+j}$$

$$\text{and } (x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{i+j}.$$

Since $r \in S_i$ and $(S_i)_{i \geq 0}$ is an α -sequence, $\alpha^j(r) \in S_{i+j}$. Similarly, $\alpha^i(s) \in S_{i+j}$, and since S_{i+j} is a subring of R , $\alpha^j(r) - \alpha^i(s)$ and $\alpha^j(r)\alpha^i(s)$ both lie in S_{i+j} . Thus, S is a subring of $A(R, \alpha)$. If S is denoted by $\Delta((S_i)_{i \geq 0})$ then it is clear that $\Delta: A \rightarrow S$ is also order-preserving.

To show that the two maps Γ and Δ are mutually inverse, let $(S_i)_{i \geq 0}$ be an α -sequence of subrings of R , and let S be a subring of $A(R, \alpha)$.

Denote the α -sequence $\Gamma(\Delta(S_i)_{i \geq 0})$ by $(T_i)_{i \geq 0}$ and let $r \in T_i$ for some $i \geq 0$. Then, $x^{-i}rx^i \in \Delta(S_i)_{i \geq 0}$ i.e. $x^{-i}rx^i = x^{-j}sx^j$ for

some $j \geq 0$ and some $s \in S_j$. Therefore, $x^{-(i+j)} \alpha^j(r) x^{i+j} = x^{-(i+j)} \alpha^i(s) x^{i+j}$, whence $\alpha^j(r) = \alpha^i(s) \in S_{i+j}$, so that $r \in S_i$, and $T_i \subseteq S_i$. The reverse inclusion follows directly from the definition of Γ and Δ ; consequently $\Gamma_0 \Delta = \text{id}_A$.

The fact that, for any subring S of $A(R, \alpha)$, $S \subseteq \Delta(\Gamma(S))$ also follows straight from the definition. On the other hand, if $x^{-i} r x^i \in \Delta(\Gamma(S))$ then $x^{-i} r x^i = x^{-j} s x^j$ for some $j \geq 0$ and $s \in S_j$ (where $\Gamma(S) = (S_j)_{j \geq 0}$), so $x^{-i} r x^i \in S$, and $\Delta_0 \Gamma = \text{id}_S$.

2.34 Lemma:

Let S be a subring of $A(R, \alpha)$ and denote $\Gamma(S)$ by $(S_i)_{i \geq 0}$. Then

- (i) S is nil iff S_i is nil for each $i \geq 0$;
- (ii) S is nilpotent iff there exists $k \geq 0$ such that $S_i^k = 0$ for each $i \geq 0$.

Proof:

(i) If S is nil and $r \in S_i$ for some $i \geq 0$, then $x^{-i} r x^i \in S$ and there exists $n \geq 0$ such that $(x^{-i} r x^i)^n = 0$. But this means $r^n = 0$, so that S_i is nil. Similarly, if each S_i is nil and $x^{-i} r x^i \in S$, then $r \in S_i$ and is therefore nilpotent. Clearly $x^{-i} r x^i$ is also nilpotent, and S is nil.

(ii) The proof of lemma 2.27, applied to subrings rather than left ideals, works here.

2.35 Theorem:

Let R be a ring which satisfies:

- (i) Each nil subring of R is nilpotent;
- (ii) The set of indices of nilpotence of subrings of R is bounded.

Then nil subrings of $A(R, \alpha)$ are nilpotent.

Proof:

Let S be a nil subring of $A(R, \alpha)$. By lemma 2.34, $(S_i)_{i \geq 0}$ ($= \Gamma(S)$) is an α -sequence of nil subrings of R , and by condition (ii) of the theorem, there exists $k \in \mathbb{N}$ with $S_i^k = 0$ for all $i \geq 0$.

By lemma 2.34, S is a nilpotent subring of $A(R, \alpha)$.

2.36 Corollary:

If R is a left Goldie ring, then nil subrings of $A(R, \alpha)$ are nilpotent.

Proof:

By theorem 4.4 of [20], nil subrings of left Goldie rings are nilpotent, of index not greater than $k(\dim R + 1)$ where $\dim R$ is the left Goldie dimension of R and k is the index of nilpotence of the left singular ideal $Z(R)$ of R . Theorem 2.35 therefore gives the result.

CHAPTER 3

LEFT JORDAN RINGS.

The study of the ring $A(R, \alpha)$ is greatly aided if it is assumed to be left Noetherian - in other words, if R is assumed to be left Jordan. In this chapter, several aspects of $A(R, \alpha)$ are studied under that assumption.

The aim of the first section is to study the left Jordan condition itself; in particular to show that it is stable under the formation of matrix rings, polynomial rings and localizations when the new rings are equipped with appropriate injective endomorphisms. The first result of the section, easily deduced from 1.37, states that any left Artinian ring, with any monomorphism, is left Jordan, so it can be seen that the class of left Jordan rings is wide enough to warrant further study.

Localization is then examined in more detail, and in particular it is shown that if R is a commutative local ring, then so is $A(R, \alpha)$. Also, in the case where α can be extended to a monomorphism α_p on the localization R_p , where P is a left localizable prime ideal, certain conditions are found under which $A(R_p, \alpha_p)$ may be viewed as the localization of $A(R, \alpha)$ at a prime ideal.

Section 3 moves on to study Krull dimension, and relates the Krull dimension of $A(R, \alpha)$ to the set of closed left ideals of R . There follows a slight diversion, where the left Jordan condition is temporarily dropped in order to answer a question asked by Jordan in [16]:

if R is Noetherian of Krull dimension 1, does $A(R, \alpha)$ necessarily have Krull dimension? The answer is in the negative, and is provided by example 3.16.

In section four, the α -sequence of the Jacobson radical of $A(R, \alpha)$ is determined in terms of maximal closed left ideals of R , and in section five, maximal left ideals of left Artinian rings are shown to be closed.

Finally, these results are combined in order to generalize a result of Jategaonkar, which states that if R is left Artinian, then $\alpha^{-1}(J(R)) = J(R)$, for any monomorphism $\alpha: R \rightarrow R$.

51. The Left Jordan Condition.

3.1 Proposition:

If R is a left Artinian ring and $\alpha: R \rightarrow R$ is a monomorphism, then (R, α) is left Jordan.

Proof:

Since R is left Artinian, $A(R, \alpha)$ is left Artinian, by theorem 1.37. By Hopkins' theorem ([6], theorem 2.10), $A(R, \alpha)$ is left Noetherian, i.e. (R, α) is left Jordan.

Notation:

Given a ring R and a monomorphism $\alpha: R \rightarrow R$, denote by $\bar{\alpha}$ the monomorphism on $M_n(R)$ (the full $n \times n$ matrix ring over R) obtained by letting $M \in M_n(R)$ have (i, j) -entry r_{ij} , and defining $\bar{\alpha}(M)$ to be the matrix whose (i, j) -entry is $\alpha(r_{ij})$.

3.2 Theorem:

If (R, α) is left Jordan, then so is $(M_n(R), \bar{\alpha})$, for any $n \in \mathbb{N}$.

Proof:

The first step is to show that $M_n(A(R, \alpha))$ is isomorphic to $A(M_n(R), \bar{\alpha})$.

Let $B \in M_n(A(R, \alpha))$. Then there exists $i \geq 0$ and a matrix $B' \in M_n(R)$ such that

$$B = \begin{bmatrix} x^{-i} & & & \\ & x^{-i} & & \\ & & \ddots & \\ & & & x^{-i} \end{bmatrix} B' \begin{bmatrix} x^i & & & \\ & x^i & & \\ & & \ddots & \\ & & & x^i \end{bmatrix} \quad (\text{where } M_n(A(R, \alpha))$$

is regarded as a subring of $M_n(R[x, x^{-1}, \alpha])$). A matrix B' which satisfies this condition will be called a B-matrix of order i . Clearly, there is at most one B-matrix of order i , for given $i \geq 0$ and $B \in M_n(A(R, \alpha))$.

To show that a B-matrix of order i exists for some $i \geq 0$, write the (j, k) -entry of B as $B_{jk} = x^{-i_{jk}} a_{jk} x^{i_{jk}}$ where $i_{jk} \geq 0$ and $a_{jk} \in R$ for $j, k = 1, \dots, n$. If $i = \max\{i_{jk} \mid j, k = 1, \dots, n\}$ then $B_{jk} = x^{-i} a_{jk} x^i$ and

$$B = \begin{bmatrix} x^{-i} & & & \\ & x^{-i} & & \\ & & \ddots & \\ & & & x^{-i} \end{bmatrix} B' \begin{bmatrix} x^i & & & \\ & x^i & & \\ & & \ddots & \\ & & & x^i \end{bmatrix} \quad \text{where}$$

$$B'_{jk} = a_{jk}. \quad \text{So } B' \text{ is the B-matrix of order } i.$$

Now attempt to define a map $\psi: M_n(A(R, \alpha)) \rightarrow A(M_n(R), \bar{\alpha})$ by $\psi(B) = x^{-i} B_i x^i$ where $i \geq 0$ is such that the B-matrix of order i exists; B_i is the B-matrix of order i .

To show that ψ is a well defined map, let B_i and B_j be B-matrices of orders i and j respectively, so that

$$B = \begin{bmatrix} x^{-i} & & \\ & x^{-i} & \\ & & \ddots \\ & & & x^{-i} \end{bmatrix} B_i \begin{bmatrix} x^i & & \\ & x^i & \\ & & \ddots \\ & & & x^i \end{bmatrix} = \begin{bmatrix} x^{-j} & & \\ & x^{-j} & \\ & & \ddots \\ & & & x^{-j} \end{bmatrix} B_j \begin{bmatrix} x^j & & \\ & x^j & \\ & & \ddots \\ & & & x^j \end{bmatrix}.$$

Writing the (k, l) -entry of B_i as b_{kl} , and the (k, l) -entry of B_j as c_{kl} gives

$$x^{-i} b_{kl} x^i = x^{-j} c_{kl} x^j$$

and so $x^{-(i+j)} \alpha^j(b_{kl}) x^{i+j} = x^{-(i+j)} \alpha^i(c_{kl}) x^{i+j}$ for all $k, l = 1, \dots, n$. Thus, $\alpha^j(b_{kl}) = \alpha^i(c_{kl})$, therefore $\bar{\alpha}^j(B_i) = \bar{\alpha}^i(B_j)$.

Now, $x^{-i} B_i x^i = x^{-(i+j)} \bar{\alpha}^j(B_i) x^{i+j} = x^{-(i+j)} \bar{\alpha}^i(B_j) x^{i+j} = x^{-j} B_j x^j$, so that ψ is well-defined.

To see that ψ is a ring isomorphism, let $B, C \in M_n(A(R, \alpha))$, and denote their (k, l) -entries by B_{kl} and C_{kl} respectively. So $B_{kl} = x^{-i_{kl}} b_{kl} x^{i_{kl}}$ and $C_{kl} = x^{-j_{kl}} c_{kl} x^{j_{kl}}$ where $b_{kl}, c_{kl} \in R$ and i_{kl}, j_{kl} are integers for $k, l = 1, \dots, n$. Let $i = \max\{i_{kl}, j_{kl} \mid k, l = 1, \dots, n\}$.

$$\text{Then } B_{kl} = x^{-i} \alpha^{i-i_{kl}}(b_{kl}) x^i \quad (1)$$

$$\text{and } C_{kl} = x^{-i} \alpha^{i-j_{kl}}(c_{kl}) x^i. \quad (2)$$

Hence, $(B+C)_{k\ell} = x^{-i} [\alpha^{i-i_{k\ell}} (b_{k\ell}) + \alpha^{i-j_{k\ell}} (c_{k\ell})] x^i$
 and $\psi(B+C) = x^{-i} D x^i$ where $D_{k\ell} = \alpha^{i-i_{k\ell}} (b_{k\ell}) + \alpha^{i-j_{k\ell}} (c_{k\ell})$.

But from (1) and (2),

$$\psi(B) = x^{-i} D' x^i \text{ where } D'_{k\ell} = \alpha^{i-i_{k\ell}} (b_{k\ell}) \quad (3)$$

$$\text{and } \psi(C) = x^{-i} D'' x^i \text{ where } D''_{k\ell} = \alpha^{i-j_{k\ell}} (c_{k\ell}). \quad (4)$$

Thus, ψ is additive.

To show that ψ preserves the multiplicative structure, note that (1) and (2) above yield

$$(BC)_{k\ell} = x^{-i} \left[\sum_{p=1}^n \alpha^{i-i_{kp}} (b_{kp}) \alpha^{i-j_{p\ell}} (c_{p\ell}) \right] x^i,$$

which means that $\psi(BC) = x^{-i} D x^i$ where $D_{k\ell} = \sum_{p=1}^n \alpha^{i-i_{kp}} (b_{kp}) \alpha^{i-j_{p\ell}} (c_{p\ell})$

for $k, \ell = 1, \dots, n$.

But from (3) and (4) above, $\psi(B)\psi(C) = x^{-i} D' D'' x^i$ and since $D' D'' = D$, ψ is a ring homomorphism.

Now assume $\psi(B) = 0$ for $B \in M_n(A(R, \alpha))$. Then, $x^{-i} B_i x^i = 0$, where B_i is the B -matrix of order i . By definition of B_i , $B_i = 0$ implies $B = 0$. Thus ψ is injective. It is clear that ψ is surjective, so $M_n(A(R, \alpha)) \cong A(M_n(R), \bar{\alpha})$.

Now, if (R, α) is left Jordan, then $A(R, \alpha)$ is left Noetherian, and therefore so is $M_n(A(R, \alpha))$. By the isomorphism above, this means $A(M_n(R), \bar{\alpha})$ is left Noetherian, so $(M_n(R), \bar{\alpha})$ is left Jordan.

Notation:

Given a ring R and a monomorphism $\alpha: R \rightarrow R$, $\tilde{\alpha}$ will denote the monomorphism obtained on $R[y]$, the ring of polynomials in one indeterminate over R , by defining

$$\tilde{\alpha}\left(\sum_{n=0}^k f_n y^n\right) = \sum_{n=0}^k \alpha(f_n) y^n.$$

3.3 Theorem:

If (R, α) is left Jordan, then so is $(R[y], \tilde{\alpha})$.

Proof:

The first step is to show that $A(R[y], \tilde{\alpha}) \cong A(R, \alpha)[y]$.

Let $f \in A(R, \alpha)[y]$. Then $f = \sum_{k=0}^n x^{-i_k} r_k x^{i_k} y^k$ where each $r_k \in R$;

or $f = \sum_{k=0}^n x^{-i} x^{i-i_k} (r_k) x^{i_k} y^k$ where $i = \max\{i_k \mid k = 0, \dots, n\}$.

Now, regarding $A(R, \alpha)[y]$ as a subring of $R[x, x^{-1}, \alpha][y]$, f may be written as

$$f = \sum_{k=0}^n x^{-i} x^{i-i_k} (r_k) y^k x^i.$$

Thus, f can be written in the form $x^{-i} f' x^i$ where $i \geq 0$ and $f' \in R[y]$. Clearly, for given i and f , f' is unique - it will be referred to as the f -polynomial of order i .

Now, attempt to define a map $\psi: A(R, \alpha)[y] \rightarrow A(R[y], \tilde{\alpha})$ by putting $\psi(f) = x^{-i} f' x^i$ where i is such that the f -polynomial of order i exists, and f' is the f -polynomial of order i .

To show that ψ is well-defined, let f_i and f_j be the f -polynomials of order i and j respectively, and write

$$f_i = \sum_{k=0}^n a_k y^k, \quad f_j = \sum_{k=0}^m b_k y^k.$$

$$\text{Then } f = \sum_{k=0}^n (x^{-i} a_k x^i) y^k = \sum_{k=0}^m (x^{-j} b_k x^j) y^k.$$

Comparing coefficients gives $n = m$ and $x^{-i} a_k x^i = x^{-j} b_k x^j$ for each $k = 0, \dots, n$. Therefore, $x^{-(i+j)} \alpha^j(a_k) x^{i+j} = x^{-(i+j)} \alpha^i(b_k) x^{i+j}$, or $\alpha^j(a_k) = \alpha^i(b_k)$.

But this means that $\alpha^j(f_i) = \alpha^i(f_j)$, so that $x^{-i} f_i x^i = x^{-(i+j)} \alpha^j(f_i) x^{i+j} = x^{-(i+j)} \alpha^i(f_j) x^{i+j} = x^{-j} f_j x^j$, and ψ is well-defined.

To show that ψ is a ring homomorphism, let $f, g \in A(R, \alpha)[y]$, with $f = \sum_{k=0}^n (x^{-i} a_k x^i) y^k$ and $g = \sum_{k=0}^n (x^{-j} b_k x^j) y^k$. Let

$$i = \max(i_k, j_k \mid k = 0, \dots, n).$$

$$\begin{aligned} \text{Then, } f+g &= \sum_{k=0}^n (x^{-i} a_k x^i + x^{-j} b_k x^j) y^k \\ &= \sum_{k=0}^n x^{-i} (\alpha^{i-j}(a_k) + \alpha^{i-j}(b_k)) x^i y^k \end{aligned}$$

$$\text{so } \psi(f+g) = x^{-i} f_i x^i \text{ where } f_i = \sum_{k=0}^n (\alpha^{i-j}(a_k) + \alpha^{i-j}(b_k)) y^k.$$

$$\text{But since } f = \sum_{k=0}^n x^{-i} \alpha^{i-j}(a_k) x^i y^k \text{ and } g = \sum_{k=0}^n x^{-i} \alpha^{i-j}(b_k) x^i y^k,$$

$$\psi(f) + \psi(g) = \psi(f+g). \quad (1)$$

Now, to look at the multiplicative property of ψ , fg can be

$$\begin{aligned} \text{written as } fg &= \sum_{k=0}^{2n} \left[\sum_{p+q=k} (x^{-i} p a_p x^i) (x^{-j} q b_q x^j) \right] y^k \\ &= \sum_{k=0}^{2n} x^{-i} \left[\sum_{p+q=k} \alpha^{i-i} p (a_p)^\alpha^{i-j} q (b_q) \right] x^i y^k \end{aligned}$$

so that $\psi(fg) = x^{-i} h x^i$ where h is the polynomial given by

$$h = \sum_{k=0}^{2n} \sum_{p+q=k} \alpha^{i-i} p (a_p)^\alpha^{i-j} q (b_q) y^k.$$

From (1) it can be seen that $\psi(f) = x^{-i} f_i x^i$ where

$$f_i = \sum_{k=0}^n \alpha^{i-i} k (a_k) y^k; \text{ and that } \psi(g) = x^{-j} g_j x^j \text{ where } g_j = \sum_{k=0}^n \alpha^{i-j} k (b_k) y^k.$$

Since $f_i g_j = h$, it is evident that $\psi(fg) = \psi(f)\psi(g)$, so that ψ is a ring homomorphism. It is clear that ψ is bijective.

Thus, $A(R[y], \tilde{\alpha}) \cong A(R, \alpha)[y]$.

Now, if (R, α) is left Jordan, then $A(R, \alpha)$ is left Noetherian and by the Hilbert basis theorem ([22], Vol. I, theorem 1, p. 202) so is $A(R, \alpha)[y]$. By the above isomorphism, this means that $A(R[y], \tilde{\alpha})$ is left Noetherian, i.e. $(R[y], \tilde{\alpha})$ is left Jordan.

Next, the behaviour of the left Jordan condition is examined when R is localized at an appropriate prime ideal.

To be specific, let R be a prime, left Noetherian ring and let P be a left localizable prime ideal of R which satisfies $\alpha(C(P)) \subseteq C(P)$.

Then, as in theorem 1.3, the set

$$I = \{r \in R \mid cr = 0 \text{ for some } c \in C(P)\}$$

is an ideal of R , and since R is prime, I is essential as a left ideal if it is non-zero. In that case, by Goldie's theorem, I contains a regular element, which is clearly impossible.

Thus $I = 0$ and α may be extended to a monomorphism α_p on the localization R_p by defining $\alpha_p(c^{-1}r) = \alpha(c)^{-1}\alpha(r)$ where $r \in R$, $c \in C(P)$.

3.4 Theorem:

Let R be a prime, left Noetherian ring with a left localizable prime ideal P which satisfies $\alpha(C(P)) \subseteq C(P)$.

Then, if (R, α) is left Jordan, so is (R_p, α_p) .

Proof:

Let S denote the set $\bigcup_{i \geq 0} x^{-i}C(P)x^i \subseteq A(R, \alpha)$. Since $\alpha(C(P)) \subseteq C(P)$, and since, as in theorem 1.3, $C(P)$ consists of regular elements of R , proposition 1.25 shows that S consists of regular elements of $A(R, \alpha)$. The fact that $\alpha(C(P)) \subseteq C(P)$ also ensures that S is a multiplicatively closed set.

Let $x^{-i}rx^i \in A(R, \alpha)$ and $x^{-j}cx^j \in S$.

Since $C(P)$ is a left Ore subset of R , and since $\alpha(C(P)) \subseteq C(P)$, there exist non-zero elements $r_1 \in R$, $c_1 \in C(P)$ such that $r_1\alpha^i(c) = c_1\alpha^j(r)$.

But this means that $x^{-(i+j)} r_1 \alpha^i (c) x^{i+j} = x^{-(i+j)} c_1 \alpha^j (r) x^{i+j}$,
and therefore $(x^{-(i+j)} r_1 x^{i+j}) x^{-j} c x^j = (x^{-(i+j)} c_1 x^{i+j}) x^{-i} r x^i$.

Since $x^{-(i+j)} c_1 x^{i+j} \in S$, $A(R, \alpha)$ has the left Ore condition with respect to S , and it is therefore possible to form the left localization $S^{-1}A(R, \alpha)$.

Now consider the ring $A(R_p, \alpha_p)$. Since R is a subring of R_p and $\alpha = \alpha_p|_R$, $A(R, \alpha)$ is a subring of $A(R_p, \alpha_p)$.

If $x^{-i} c x^i \in S$, then $c^{-1} \in R_p$ and $(x^{-i} c^{-1} x^i)(x^{-i} c x^i) = (x^{-i} c x^i)(x^{-i} c^{-1} x^i) = 1$, so that each element of S is a unit of $A(R_p, \alpha_p)$.

Furthermore, any element of $A(R_p, \alpha_p)$ has the form $x^{-i} c^{-1} r x^i$ where $c \in C(P)$ and $r \in R$. But $x^{-i} c^{-1} r x^i = (x^{-i} c x^i)^{-1} x^{-i} r x^i$.

Thus, $A(R_p, \alpha_p)$ may be identified with the localization $S^{-1}A(R, \alpha)$.

Now, if $A(R, \alpha)$ is left Noetherian, then so is $S^{-1}A(R, \alpha)$ and therefore so is $A(R_p, \alpha_p)$. Hence, (R_p, α_p) is left Jordan.

3.5 Remark:

The same method as that used in the proof of theorem 3.4 can be used in the more general case where R is any ring with an α -invariant, multiplicatively closed subset T of regular elements, which satisfies the left Ore condition. In this case, α again defines a monomorphism $\alpha_T: R_T \rightarrow R_T$, and if (R, α) is left Jordan, then so is (R_T, α_T) .

§2. Localization.

In view of theorem 3.4, the question arises as to whether, given an appropriate localizable prime ideal P , $A(R_P, \alpha_P)$ is a local ring, and if it is, how it is related to $A(R, \alpha)$.

The first result of this section shows that if R is a commutative local ring, then $A(R, \alpha)$ is always local, and then the problem of relating $A(R_P, \alpha_P)$ to $A(R, \alpha)$ is solved in the case where R is a commutative domain: in fact, $A(R_P, \alpha_P)$ can be viewed as a localization of $A(R, \alpha)$ at a particular prime ideal. For the definitions and basic properties of commutative localization, refer to chapter 3 of [1].

Finally, an analogue of these results is proved when R is a prime, left Noetherian, left Jordan ring.

3.6 Proposition:

Let R be a commutative local ring, $\alpha: R \rightarrow R$ a monomorphism. Then $A(R, \alpha)$ is also a local ring.

Proof:

Let $x^{-i_k} r_k x^{i_k} \in A(R, \alpha)$ for $k = 1, \dots, n$ and assume that $\sum_{k=1}^n x^{-i_k} r_k x^{i_k}$ is a unit of $A(R, \alpha)$.

Thus, if $i = \max\{i_k \mid k = 1, \dots, n\}$ then $x^{-i} (\sum_{k=1}^n \alpha^{i-i_k}(r_k)) x^i$ is a unit of $A(R, \alpha)$ and by proposition 1.27, there exists $m \geq 0$ with $\alpha^m (\sum_{k=1}^n \alpha^{i-i_k}(r_k)) = \sum_{k=1}^n \alpha^{m+i-i_k}(r_k)$ a unit of R . If none of the

$\alpha^{m+i-i_k}(r_k)$ is a unit of R , then $\alpha^{m+i-i_k}(r_k)R \subseteq M$ where M is the unique maximal ideal of R , which implies that $(\sum_{k=1}^n \alpha^{m+i-i_k}(r_k))R \subseteq M$.

Clearly this is impossible, so there exists $k_0 \in \{1, \dots, n\}$ such that

$\alpha^{m+i-i_{k_0}}(r_{k_0})$ is a unit of R .

By proposition 1.27, this means that $x^{-i_{k_0}} r_{k_0} x^{i_{k_0}}$ is a unit of $A(R, \alpha)$.

Now, let M_1 and M_2 be two distinct maximal ideals of $A(R, \alpha)$, and let $m_1 \in M_1$ but $m_1 \notin M_2$.

Then $m_1 A(R, \alpha) + M_2 = A(R, \alpha)$, so there exists $m_2 \in M_2$, $a \in A(R, \alpha)$ such that $m_1 a + m_2 = 1$. Since $m_1 a \in M_1$, the above argument shows that m_2 must be a unit, whence $M_2 = A(R, \alpha)$ and $A(R, \alpha)$ is local.

Notation:

If P is a prime ideal of a ring R , then $T(P)$ will denote the set $\{r \in R \mid \alpha^k(r) \in P \text{ for all } k \geq 0\}$.

If $S \subseteq R$, then S' will denote the set-theoretic complement of S .

3.7 Theorem:

Let R be a commutative domain, and let P be a prime ideal of R satisfying $\alpha(P') \subseteq P'$. Then

- (i) $T(P)$ is an α -invariant prime ideal of R ;
- (ii) $G(T(P))$ is a prime ideal of $A(R, \alpha)$;
- (iii) $A(R_p, \alpha_p) \cong A(R, \alpha)_{G(T(P))}$.

Proof:

(i) It is clear that $T(P)$ is an α -invariant ideal of R .
To show that it is prime, let $a, b \in R$ with $ab \in T(P)$, and assume that $a \notin T(P)$. Then, there exists $k \geq 0$ with $\alpha^k(a) \notin P$, and since $\alpha(P') \subseteq P'$, this means $\alpha^n(a) \notin P$ for all $n \geq k$.

But $\alpha^n(a)\alpha^n(b) = \alpha^n(ab) \in P$, whence $\alpha^n(b) \in P$ for all $n \geq k$.
Since $\alpha(P') \subseteq P'$, $\alpha^n(b) \in P$ for all $0 \leq n \leq k$, so $b \in T(P)$ and $T(P)$ is prime.

(ii) Let S denote the multiplicatively closed subset
 $\bigcup_{i \geq 0} x^{-i} P' x^i$ of $A(R, \alpha)$.

Then $x^{-i} r x^i \in S$ iff there exists $k \geq 0$ such that $\alpha^k(r) \notin P$.
Indeed, assume that $x^{-i} r x^i \in S$ for some $i \geq 0$, $r \in R$. Then, for some $j \geq 0$,

$$x^{-i} r x^i = x^{-j} s x^j \quad \text{where } s \in P' ,$$

i.e. $x^{-(i+j)} \alpha^j(r) x^{i+j} = x^{-(i+j)} \alpha^i(s) x^{i+j}$, so that $\alpha^j(r) = \alpha^i(s)$.
Since $\alpha(P') \subseteq P'$, $\alpha^j(r) \notin P$.

On the other hand, if $\alpha^k(r) \in P'$ then, for any $i \geq 0$,
 $x^{-(i+k)} \alpha^k(r) x^{i+k} \in S$, i.e. $x^{-i} r x^i \in S$.

$$\begin{aligned} \text{Thus, } S' &= \{x^{-i}rx^i \mid i \geq 0 ; \alpha^k(r) \in P \text{ for all } k \geq 0\} \\ &= G(T(P)) . \end{aligned}$$

Since $S = G(T(P))'$ is multiplicatively closed, $G(T(P))$ is a prime ideal.

(iii) Since S is a multiplicatively closed set, it is possible to form the localization $S^{-1}A(R, \alpha) \cong A(R, \alpha)_{G(T(P))}$.

Now, regarding $A(R, \alpha)$ as a subring of $A(R_p, \alpha_p)$, it is evident that each element $x^{-i}cx^i$ of S ($c \in P'$) has an inverse, $x^{-i}c^{-1}x^i$, in $A(R_p, \alpha_p)$, and furthermore, any element $x^{-i}c^{-1}rx^i$ of $A(R_p, \alpha_p)$ ($r \in R, c \in P'$) can be written in the form $(x^{-i}cx^i)^{-1}(x^{-i}rx^i)$.

By theorem 1.1, $A(R_p, \alpha_p) \cong A(R, \alpha)_{G(T(P))}$.

3.8 Example:

Let $R = K[y]$ where K is a field and y is an indeterminate, and let α be the K -endomorphism of R such that $\alpha(y) = y^2$.

Let P be the prime ideal generated by y . Clearly $\alpha(P') \subseteq P'$, and in this case $\alpha(P) \subseteq P$ so that $T(P) = P$. By proposition 3.6, $A(K[y]_P, \alpha_p)$ is a local ring and by theorem 3.7, it is isomorphic to $A(K[y], \alpha)_{G(P)}$.

In the case where R and $A(R, \alpha)$ are not necessarily commutative, the following result provides an analogue of theorem 3.7.

3.9 Theorem:

Let R be a prime, left Noetherian, left Jordan ring with an

α -stable, left localizable prime ideal P .

Then (i) $\alpha(C(P)) \subseteq C(P)$;

(ii) $G(P)$ is a left localizable prime ideal of $A(R, \alpha)$;

(iii) $A(R_P, \alpha_P) \cong A(R, \alpha)_{G(P)}$.

Proof:

(i) Since P is an α -stable ideal, α may be used to define a monomorphism $\bar{\alpha} : R/P \rightarrow R/P$, by putting $\bar{\alpha}(r+P) = \alpha(r) + P$. Now, if $r \in C(P)$, then $r+P$ is regular in R/P . But R/P has a simple Artinian left quotient ring, so by proposition 2.4 of [13], $\bar{\alpha}(r+P) = \alpha(r)+P$ is also regular in R/P . Thus $\alpha(r) \in C(P)$.

(ii) To see that $G(P)$ is a prime ideal, consider the α -sequence $\Gamma(G(P))$, and denote it by $(I_i)_{i \geq 0}$. Since $G(P)$ is an α -stable ideal, $I_i = I_j$ for all $i, j \geq 0$, by proposition 2.4.

Thus, for each $i \geq 0$, $I_i = I_0 = G(P) \cap R$, but by theorem 2.2, $G(P) \cap R = \bigcup_{n \geq 0} \alpha^{-n}(P)$, and since P is α -stable, $I_i = P$ for all $i \geq 0$.

Now let $(A_i)_{i \geq 0}$ and $(B_i)_{i \geq 0}$ be two α -sequences of closed left ideals of R such that $A_i B_i \subseteq I_i$ for each $i \geq 0$. That is, $A_i B_i \subseteq P$ and since P is a prime ideal, either $A_i \subseteq P$ or $B_i \subseteq P$ for all $i \geq 0$.

Since R is left Jordan, there exists $m \geq 0$ such that, for all $i \geq m$, $\rho_1(A_i) = A_{i+1}$ and $\rho_1(B_i) = B_{i+1}$. Assume, without loss of

generality, that $A_m \subseteq P$. Then, for all $n = 0, 1, \dots, m$,
 $\alpha^{-(m-n)}(A_m) \subseteq \alpha^{-(m-n)}(P) = P$, so that $A_i \subseteq P$ for $i = 0, 1, \dots, m$.

$$\begin{aligned} \text{Also, } A_{m+k} &= \rho_k(A_m) \\ &= \bigcup_{n \geq 0} \alpha^{-n}(R\alpha^{n+k}(A_m)) \\ &\subseteq \bigcup_{n \geq 0} \alpha^{-n}(R\alpha^{n+k}(P)) \\ &\subseteq P \text{ for all } k \geq 0, \text{ since } P \text{ is } \alpha\text{-stable.} \end{aligned}$$

Therefore, $(A_i)_{i \geq 0} \leq \Gamma(G(P))$ and $\Gamma(G(P))$ is a prime α -sequence.
 By theorem 2.11, $G(P)$ is a prime ideal of $A(R, \alpha)$.

To show that $G(P)$ is left localizable, let S be the subset
 $\bigcup_{i \geq 0} x^{-i}C(P)x^i$ of $A(R, \alpha)$, and let $x^{-i}cx^i \in S$, where $c \in C(P)$.

If $x^{-j}rx^j \in A(R, \alpha)$ is such that $(x^{-j}rx^j)(x^{-i}cx^i) \in G(P)$, then

$$x^{-(i+j)}\alpha^i(r)\alpha^j(c)x^{i+j} \in G(P)$$

and by theorem 2.2, $\alpha^i(r)\alpha^j(c) \in K_0 G(P) = P$.

But $\alpha(C(P)) \subseteq C(P)$, so $\alpha^j(c) \in C(P)$, whence $\alpha^i(r) \in P$.
 Thus $x^{-(i+j)}\alpha^i(r)x^{i+j} \in G(P)$, or $x^{-j}rx^j \in G(P)$.

So, $S \subseteq {}^lC(G(P))$, and a similar argument on the right gives
 $S \subseteq C^r(G(P))$, so that $S \subseteq C(G(P))$.

On the other hand, let $x^{-i}cx^i \in C(G(P))$ and assume that $rc \in P$
 for some $r \in R$.

Then, $(x^{-i}rx^i)(x^{-i}cx^i) \in G(P)$ for any $i \geq 0$,

i.e. $x^{-i}rx^i \in G(P)$ and therefore

$r \in K_0G(P) = P$, again by theorem 2.2.

Thus, $C(G(P)) \subseteq \bigcup_{i \geq 0} x^{-i} C(P) x^i$, and a similar right-handed argument gives $C(G(P)) \subseteq \bigcup_{i \geq 0} x^{-i} C'(P) x^i$, whence $S = C(G(P))$.

The proof of theorem 3.4 shows that $A(R, \alpha)$ has the left Ore condition with respect to S , so $G(P)$ is left localizable.

(iii) The proof of theorem 3.4 also shows that S is a multiplicatively closed set of regular elements of $A(R, \alpha)$, so it is possible to form the left localization $S^{-1}A(R, \alpha)$, which has been shown in (ii) above to be the same as $A(R, \alpha)_{G(P)}$.

But as in the proof of theorem 3.4, $S^{-1}A(R, \alpha)$ may be identified with $A(R_p, \alpha_p)$, so that $A(R_p, \alpha_p) \cong A(R, \alpha)_{G(P)}$.

Remark:

The isomorphism given in theorem 3.9 (iii) above shows that $A(R_p, \alpha_p)$ is a local ring, since $G(P)$ is prime and left localizable.

3.10 Example:

Let K be a field, $\sigma: K \rightarrow K$ a monomorphism which is not surjective, and define $\sigma_1: K[y] \rightarrow K[y]$ by

$$\sigma_1(f_n y^n + f_{n-1} y^{n-1} + \dots + f_1 y + f_0) = \sigma(f_n) y^n + \sigma(f_{n-1}) y^{n-1} + \dots + \sigma(f_0).$$

Let $R = M_2(K[y])$ and define $\alpha: R \rightarrow R$ by

$$\alpha \begin{bmatrix} f(y) & g(y) \\ h(y) & k(y) \end{bmatrix} = \begin{bmatrix} \sigma_1(f(y)) & \sigma_1(g(y)) \\ \sigma_1(h(y)) & \sigma_1(k(y)) \end{bmatrix}.$$

Since $A(K, \sigma)$ is a field (by proposition 1.27), (K, σ) is left Jordan, and by theorem 3.3, so is $(K[y], \sigma_1)$. By theorem 3.2, R is left Jordan too. R is also prime and left Noetherian.

Let $\langle y \rangle$ denote the ideal of $K[y]$ generated by y , and let $P = \begin{bmatrix} \langle y \rangle & \langle y \rangle \\ \langle y \rangle & \langle y \rangle \end{bmatrix}$. Then P is a prime ideal of R , $\alpha(P) \subseteq P$, and since σ is a monomorphism, $\alpha^{-1}(P) \subseteq P$, so that P is α -stable.

Recall that an ideal I of a ring R is said to have the left AR-property if for each left ideal K of R there exists $n \in \mathbb{N}$ with $K \cap I^n \subseteq IK$. The ideal I is said to have the right AR-property if for each right ideal K of R , there exists $n \in \mathbb{N}$ with $K \cap I^n \subseteq KI$. If each ideal of R has both the left and right AR-property, then R is called an AR-ring.

By corollary 11.8 of [3], any commutative Noetherian ring (in this case, $K[y]$) is an AR-ring, and it is well-known ([3], Corollary 11.6) that a full matrix ring over an AR-ring is again an AR-ring. So $M_2(K[y])$ is an AR-ring, and by a theorem due to P.F. Smith ([3], corollary 11.12), which states that any semiprime ideal of a left and

right Noetherian AR-ring is localizable, P is localizable.

Theorem 3.9 therefore applies, and gives $A(R_p, \alpha_p) \cong A(R, \alpha)_{G(P)}$.

§3. Krull Dimension.

Even in the case where R is assumed to be left Jordan, Krull dimension does not behave well on passage from R to $A(R, \alpha)$. This is illustrated by example 3.14, which shows that it is possible for R to have Krull dimension i (for any non-negative integer i), but for $A(R, \alpha)$ to be a field, and therefore to have Krull dimension zero. This example first appeared in Jordan's paper [16], and the object of the first part of this section is to explain these examples by relating the left Krull dimension of $A(R, \alpha)$ to the set of closed left ideals of R , in the case where R is left Jordan.

The second part of the section is really a diversion, in the sense that the left Jordan assumption on R is temporarily discarded. It is devoted to answering the following question, which was posed by Jordan in [16].

If R has left Krull dimension zero, then it is left Artinian, so by theorem 1.37, $A(R, \alpha)$ is left Artinian and therefore also has left Krull dimension zero.

However, example 3.15 shows that it is possible to have a commutative, Noetherian domain R of Krull dimension 2, with $A(R, \alpha)$ not having Krull dimension at all.

The question to be answered, then, is what happens when $\dim_R R = 1$?

Example 3.16 answers this by giving a commutative, Noetherian domain R of Krull dimension 1, and a monomorphism $\alpha: R \rightarrow R$, such that $A(R, \alpha)$ does not have Krull dimension.

For the definition and basic properties of Krull dimension, refer to [10].

3.11 Proposition:

Let C denote the collection of closed left ideals of R , and define a relation on $C \times \mathbb{N}$ by $(I, n) \sim (J, m)$ iff $\rho_m(I) = \rho_n(J)$.

Then \sim is an equivalence relation on $C \times \mathbb{N}$.

Proof:

It is clear that \sim is reflexive and symmetric. To see that it is transitive, assume that $(I, n) \sim (J, m)$ and $(J, m) \sim (K, \ell)$, so that

$$\rho_m(I) = \rho_n(J) \quad \text{and} \quad \rho_\ell(J) = \rho_m(K).$$

Applying ρ_ℓ to both sides of the first equation gives $\rho_{m+\ell}(I) = \rho_{n+\ell}(J)$, and applying ρ_n to the second equation gives $\rho_{\ell+n}(J) = \rho_{m+n}(K)$, by 1.35.

Thus, $\rho_{m+\ell}(I) = \rho_{m+n}(K)$ and applying α^{-m} gives $\rho_\ell(I) = \rho_n(K)$, again by 1.35. Thus $(I, n) \sim (K, \ell)$.

3.12 Proposition:

Let S denote the set of equivalence classes $C \times \mathbb{N}/\sim$, and define a relation \leq on S by

$$[(I,n)] \leq [(J,m)] \quad \text{if} \quad \rho_m(I) \subseteq \rho_n(J) .$$

Then \leq is a partial order on S .

Proof:

It must first be shown that \leq is well-defined. Assume that $(I_1, n_1) \sim (I, n)$ and $(J_1, m_1) \sim (J, m)$ with $\rho_m(I) \subseteq \rho_n(J)$. (1)

Since $\rho_n(I_1) = \rho_{n_1}(I)$, applying ρ_{m_1+m} gives

$$\rho_{n+m_1+m}(I_1) = \rho_{n_1+m_1+m}(I) . \quad (2)$$

Applying $\rho_{m_1+n_1}$ to each side of (1) gives

$$\rho_{m+m_1+n_1}(I) \subseteq \rho_{n+m_1+n_1}(J) . \quad (3)$$

Since $\rho_m(J_1) = \rho_{m_1}(J)$, applying ρ_{n_1+n} gives

$$\rho_{m+n_1+n}(J_1) = \rho_{m_1+n_1+n}(J) . \quad (4)$$

Now, (2), (3) and (4) yield $\rho_{n+m_1+m}(I_1) \subseteq \rho_{m+n_1+n}(J_1)$, and applying $\alpha^{-(m+n)}$ to this gives, by proposition 1.35, $\rho_{m_1}(I_1) \subseteq \rho_{n_1}(J_1)$.

Thus \leq is well-defined.

It is clear that \leq is reflexive.

To see that \leq is antisymmetric, assume that $[(I,n)] \leq [(J,m)]$ and $[(J,m)] \leq [(I,n)]$. Then $\rho_m(I) \subseteq \rho_n(J)$ and $\rho_n(J) \subseteq \rho_m(I)$, i.e. $\rho_n(J) = \rho_m(I)$ and $[(I,n)] = [(J,m)]$.

Transitivity may be proved in the same way as it was for \sim in proposition 3.11.

3.13 Theorem:

Let R be a left Jordan ring.

Then $K \dim_{A(R,\alpha)} A(R,\alpha) = K \dim S$, where S is equipped with the partial order \leq .

Proof:

Let $[(I,n)] \in S$ and define an α -sequence $(I_j)_{j \geq 0}$ of closed left ideals as follows:

$$I_j = \begin{cases} \alpha^{-(n-j)}(I) & j < n \\ I & j = n \\ \rho_{j-n}(I) & j > n. \end{cases}$$

Note that $\alpha^{-(n-j)}(I)$ is closed for $j < n$, and $\rho_{j-n}(I)$ is closed for $j > n$ since $(I_j)_{j \geq 0}$ is an α -sequence of left ideals, by proposition 1.35.

Thus, $(I_j)_{j \geq 0}$ is a stable α -sequence of closed left ideals with $I_n = I$, so that $\Delta((I_j)_{j \geq 0})$ is a left ideal of $A(R, \alpha)$. Denote the left ideal obtained in this manner by $\psi[(I, n)]$, so that ψ is a map from S to L , the lattice of left ideals of $A(R, \alpha)$.

ψ is order-preserving, because if $[(I, n)] \leq [(J, m)]$ then $\rho_m(I) \subseteq \rho_n(J)$, but $\psi[(I, n)]_{m+n} = \rho_m(I)$, and $\psi[(J, m)]_{m+n} = \rho_n(J)$, so that $\psi[(I, n)]_{m+n} \subseteq \psi[(J, m)]_{m+n}$, where $(\psi[(J, m)]_i)_{i \geq 0}$ denotes $r(\psi[(J, m)])$.

Now, for $0 \leq j < m+n$,

$$\begin{aligned} \psi[(I, n)]_{m+n-j} &= \alpha^{-j}(\psi[(I, n)]_{m+n}) \\ &\subseteq \alpha^{-j}(\psi[(J, m)]_{m+n}) = \psi[(J, m)]_{m+n-j}; \end{aligned}$$

and for all $j \geq 1$,

$$\begin{aligned} \psi[(I, n)]_{m+n+j} &= \rho_j(\psi[(I, n)]_{m+n}) \\ &\subseteq \rho_j(\psi[(J, m)]_{m+n}) = \psi[(J, m)]_{m+n+j}. \end{aligned}$$

By theorem 1.33, then, $\psi[(I, n)] \subseteq \psi[(J, m)]$.

Since any α -sequence of closed left ideals of R is stable, ψ is surjective.

To see that ψ is injective, let I, J be closed left ideals of R such that $\psi[(I, n)] = \psi[(J, m)]$ for some $n, m \geq 0$. Then by

1.33, $\psi[(I,n)]_i = \psi[(J,m)]_i$ for all $i \geq 0$, and in particular, $\psi[(I,n)]_{m+n} = \psi[(J,m)]_{m+n}$ gives $\rho_m(I) = \rho_n(J)$, so that $[(I,n)] = [(J,m)]$.

The inverse bijection, ψ^{-1} , is also order-preserving. Indeed, let I, J be left ideals of $A(R, \alpha)$ with $I \subseteq J$. Since R is left Jordan, both α -sequences $(I_i)_{i \geq 0}$ and $(J_i)_{i \geq 0}$ are stable. Let $n \geq 0$ be such that, for all $i \geq n$, $\rho_1(I_i) = I_{i+1}$ and $\rho_1(J_i) = J_{i+1}$.

Then $\psi[(I,n)] = I$ and $\psi[(J,n)] = J$.

But by theorem 1.33, $I_n \subseteq J_n$, hence $\rho_n(I_n) \subseteq \rho_n(J_n)$, and $[(I,n)] \leq [(J,n)]$, so ψ^{-1} is order-preserving.

Thus, $K \dim_{A(R, \alpha)} A(R, \alpha) = K \dim S$.

3.14 Example: (Jordan, [16])

Let K be a field, and denote by R the polynomial ring $K[x_i]_{i \geq 1}$ in a countable number of commuting indeterminates.

Denote by R_i the localization of R at the multiplicatively closed subset $K[x_j]_{j \geq i+1} - \{0\}$ and let $\alpha_i: R_i \rightarrow R_i$ be the K -endomorphism such that $\alpha_i(x_j) = x_{j+1}$ for all $j \geq 1$. If S_i denotes the quotient field of $K[x_j]_{j \geq i+1}$, then R_i may be identified with $S_i[x_1, \dots, x_i]$, which by proposition 9.2 of [10], has Krull dimension i .

But, for each $i \geq 0$, R_i does not have a proper closed ideal. So $C \times \mathbb{N} = \{0, R_i\} \times \mathbb{N}$, and clearly for any $n, m \in \mathbb{N}$, $\rho_m(0) = \rho_n(0)$

and $\rho_n(R_i) = \rho_m(R_i)$. Thus S consists of only two elements, and therefore has Krull dimension zero. By theorem 3.13, $K \dim A(R_i, \alpha_i) = 0$ for each $i \geq 0$.

Remark:

Turning away now from the left Jordan setting, the following example shows that it is possible to have a ring R of Krull dimension 2, and a monomorphism $\alpha: R \rightarrow R$ such that $A(R, \alpha)$ does not have Krull dimension. By proposition 3.1, left Krull dimension zero (that is to say, the ring R is left Artinian) is preserved on passage from R to $A(R, \alpha)$, and example 3.16 fills in the gap, first noted by Jordan in [16], at Krull dimension 1 by showing that it is possible for R to have Krull dimension 1 but for $A(R, \alpha)$ not to have Krull dimension.

3.15 Example: (Jordan, [16])

Let K be a field and let $R = K[y]$, the polynomial ring over K in one indeterminate. Let $\alpha: R \rightarrow R$ be the K -monomorphism such that $\alpha(y) = y^2$.

Now consider the ring $A(R, \alpha)[t]$ where t is an indeterminate, and extend α to $R[t]$ by defining $\tilde{\alpha}(\sum_{i=1}^n f_i t^i) = \sum_{i=1}^n \alpha(f_i) t^i$. By the proof of theorem 3.3, $A(R, \alpha)[t] \cong A(R[t], \tilde{\alpha})$, and since by 1.41 $A(R, \alpha)$ is not Noetherian, by proposition 9.1 of [10], $A(R, \alpha)[t]$ cannot have Krull dimension.

By proposition 9.2 of [10], $K \dim R[t] = 2$, but $A(R[t], \tilde{\alpha})$ does not have Krull dimension.

3.16 Example:

As in the previous example, let $R = K[y]$ and let $\alpha: R \rightarrow R$ be the K -monomorphism such that $\alpha(y) = y^2$. Note that R is a commutative, Noetherian domain of Krull dimension 1. It will be shown that $A(R, \alpha)$ does not have Krull dimension.

Let $\langle y^n \rangle$ denote the ideal of R generated by y^n , and let n be an even integer. Then clearly $\alpha(\langle y^{n/2} \rangle) \subseteq \langle y^n \rangle$. On the other hand, if $f \in R$ is such that $\alpha(f) \in \langle y^n \rangle$, then $\alpha(f) = y^n (\sum_i c_i y^i)$ where $c_i \in K$ and each i is even (since $n+i$ has to be even). Thus $f = y^{n/2} (\sum_i c_i y^{i/2}) \in \langle y^{n/2} \rangle$, and therefore $\alpha^{-1}(\langle y^n \rangle) = \langle y^{n/2} \rangle$. - (1)

Now let n be an odd integer. Then clearly $\alpha(\langle y^{\frac{n+1}{2}} \rangle) \subseteq \langle y^{n+1} \rangle \subseteq \langle y^n \rangle$. But if $f \in R$ is such that $\alpha(f) \in \langle y^n \rangle$, then $\alpha(f) = y^n (\sum_i c_i y^i)$. Again, each $n+i$ must be even, and since n is odd, each i must be odd. Thus, $\alpha(f) = y^{n+1} (\sum_i c_i y^{i-1})$, so that $f = y^{\frac{n+1}{2}} (\sum_i c_i y^{\frac{i-1}{2}}) \in \langle y^{\frac{n+1}{2}} \rangle$.

This, combined with (1) above, yields, for any $n \in \mathbb{N}$:

$$\alpha^{-1}(\langle y^n \rangle) = \begin{cases} \langle y^{n/2} \rangle & n \text{ even} \\ \langle y^{\frac{n+1}{2}} \rangle & n \text{ odd} \end{cases} \quad - (2)$$

Next, let $k \in \mathbb{N}_0$ and let Z_k denote the set

$$Z_k = \{(n_0, n_1, \dots, n_k) \in \mathbb{N}^{k+1} \mid n_0 = 1 \text{ and for all } 1 \leq i \leq k, \\ \text{either } n_i = 2n_{i-1} \text{ or } n_i = 2n_{i-1} - 1\}.$$

Define, for each $k \in \mathbb{N}$, a map $f_k: \mathbb{N} \rightarrow \mathbb{N}$ by

$$f_1(n) = 2n-1, \text{ and } f_k(n) = 2f_{k-1}(n)-1 \text{ for } k \geq 2.$$

Finally, for $k \in \mathbb{N}_0$ and $N \in Z_k$, put

$$(B_{N,k})_i = \begin{cases} \langle y^{n_i} \rangle & \text{for } 0 \leq i \leq k \\ \langle y^{f_{i-k}(n_k)} \rangle & \text{for } i \geq k+1 \end{cases}$$

where $N = (n_0, n_1, \dots, n_k)$.

By (2), $((B_{N,k})_i)_{i \geq 0}$ is an α -sequence of ideals, and by the argument prior to definition 1.31, each $(B_{N,k})_i$ is closed. By theorem 1.33, it therefore defines an ideal of $A(R, \alpha)$, which will be denoted by $B_{N,k}$.

The collection $\{B_{N,k} \mid k \in \mathbb{N}_0, N \in Z_k\}$ of ideals of $A(R, \alpha)$ will be denoted by X .

It is now claimed that given $B_{N,k}, B_{N_1,k_1} \in X$ with $B_{N,k} \not\subseteq B_{N_1,k_1}$, there exists an infinite descending chain of ideals

in X between $B_{N,k}$ and B_{N_1,k_1} .

Indeed, since $B_{N,k} \neq B_{N_1,k_1}$, there exists $m \geq k+k_1$ such that $(B_{N,k})_m \neq (B_{N_1,k_1})_m$. (Theorem 1.33 shows that $(B_{N,k})_i = (B_{N_1,k_1})_i$ for all $i \geq 0$, and if no such m exists, then $(B_{N,k})_i = (B_{N_1,k_1})_i$ for all $i \geq k+k_1$, so that $\alpha^{-j}((B_{N,k})_i) = \alpha^{-j}((B_{N_1,k_1})_i)$ for all $0 \leq j \leq i$; yielding $B_{N,k} = B_{N_1,k_1}$).

Assume that $(B_{N,k})_m = \langle y^{m_0} \rangle$ and $(B_{N_1,k_1})_m = \langle y^{m_1} \rangle$.

Since $m \geq k + k_1$, $(B_{N,k})_{m+1} = \langle y^{2m_0-1} \rangle$ and $(B_{N_1,k_1})_{m+1} = \langle y^{2m_1-1} \rangle$.

Define

$$(B_{N_2,k_2})_i = \begin{cases} (B_{N_1,k_1})_i & \text{for } 0 \leq i \leq m \\ \langle y^{2m_1} \rangle & \text{for } i = m+1 \\ \langle y^{f_{i-m-1}(2m_1)} \rangle & \text{for } i \geq m+2, \end{cases}$$

so that $k_2 = m+1$ and $N_2 \in Z_{k_2}$ has j^{th} entry n_j , such that $(B_{N_2,k_2})_j = \langle y^{n_j} \rangle$.

Since $m_0 \geq m_1 + 1$, $2m_1 - 1 < 2m_1 < 2m_0 - 1$ so that

$(B_{N,k})_{m+1} \subsetneq (B_{N_2,k_2})_{m+1} \subsetneq (B_{N_1,k_1})_{m+1}$, and because f_k is an increasing function for each $k \in \mathbb{N}$,

$$(B_{N,k})_i \subsetneq (B_{N_2,k_2})_i \subsetneq (B_{N_1,k_1})_i \text{ for all } i \geq m+2.$$

Hence, $B_{N,k} \subsetneq B_{N_2,k_2} \subsetneq B_{N_1,k_1}$.

The process can be repeated for $B_{N,k} \subsetneq B_{N_2,k_2}$, and repeated application yields the required infinite chain of ideals in X between $B_{N,k}$ and B_{N_1,k_1} .

Now assume that $A(R,\alpha)$ has Krull dimension. Then by lemma 1.1 of [10], the $A(R,\alpha)$ -module I/J has Krull dimension, for any ideals $I \supsetneq J$ of $A(R,\alpha)$.

Let $I, J \in X$ be such that $I \supsetneq J$ and $K \dim I/J = \min\{K \dim A/B \mid A \supsetneq B, A, B \in X\}$. As shown above, there exists an infinite descending chain $(I_j)_{j \geq 0}$ of ideals in X with $J \subsetneq I_j \subsetneq I$ for each $j \geq 0$.

By the definition of Krull dimension (as in [10]), there must exist $k \geq 0$ such that for all $j \geq k$, $K \dim(I_j/I_{j+1}) < K \dim I/J$.

This is a contradiction, so $A(R,\alpha)$ cannot have Krull dimension.

54. The Jacobson Radical.

The behaviour of the Jacobson radical on passing from R to $A(R,\alpha)$ is not at all straightforward. For instance, examples 3.17 and 3.18

below show that it is possible for either of the rings R or $A(R, \alpha)$ to be semiprimitive, but not the other. However, it is then shown in proposition 3.19 that if $J(R)$ is α -invariant, which is always the case when R is left Artinian (by lemma 1.1 of [13]), then $A(R, \alpha)$ semiprimitive implies that R is semiprimitive.

Attention is then turned to the Jacobson radical of $A(R, \alpha)$ under the assumption that R is left Jordan, with the object of calculating its α -sequence in terms of closed left ideals of R . The method used is similar to that employed in the previous chapter to study the nilpotent radical of $A(R, \alpha)$.

First, those α -sequences which give rise to maximal left ideals of $A(R, \alpha)$ are determined, and then it is possible to find the α -sequence of the intersection of all the maximal left ideals - i.e. the α -sequence of the Jacobson radical. In fact, if $(J_i)_{i \geq 0}$ denotes the α -sequence of $J(A(R, \alpha))$, then J_i is given by the intersection of all the maximal closed left ideals of R .

Finally, this enables some of the behaviour of semiprimitivity on passage from R to $A(R, \alpha)$ to be explained.

3.17 Example: (Jordan, [16])

Let B be the formal power series ring $K[[x_i]]_{i \geq 1}$ in a countable set of commuting indeterminates over a field K , and let R be the commutative polynomial ring $B[x_0]$.

Define $\alpha: R \rightarrow R$ to be the K -algebra endomorphism such that $\alpha(x_i) = x_{i+1}$ for all $i \geq 0$, and let I be the ideal of R generated by $\{x_i\}_{i \geq 0}$. Then $\alpha^{-1}(I) = I$, and $\alpha(I) \subseteq I$, so for all $a \in I$, $1 - \alpha(a)$ is a unit of R (by, for instance, theorem 2, p.131 of vol.II of [22]).

Now consider the α -sequence $(I_i)_{i \geq 0}$ where $I_i = I$ for all $i \geq 0$. If $x^{-i}rx^i \in \Delta((I_i)_{i \geq 0})$, then $r \in I$ and

$$1 - x^{-i}rx^i = x^{-(i+1)}(1 - \alpha(r))x^{i+1}, \quad \text{so by}$$

proposition 1.37, $1 - x^{-i}rx^i$ is a unit of $A(R, \alpha)$. Thus $\Delta((I_i)_{i \geq 0})$ is a quasi-regular ideal of $A(R, \alpha)$, so $A(R, \alpha)$ is not semiprimitive.

In general, by theorem 4, p. 12 of [11], for a ring R which has no nil ideals, $J(R[t]) = 0$. Here, B is a domain, so $B[x_0] = R$ is semiprimitive.

3.18 Example:

Let S be the polynomial ring $K[x_i]_{i \in \mathbb{Z}}$ in an infinite number of commuting indeterminates over a field K , and let $\bar{\alpha}: S \rightarrow S$ be the K -monomorphism such that $\bar{\alpha}(x_i) = x_{i+1}$. Let P be the ideal generated by $\{x_{-i} \mid i \geq 1\}$, i.e. $P = \sum_{i=1}^{\infty} x_{-i} S$. Then P is a prime ideal of S .

Let R denote the localization of S at P . If $f \in S - P$, then $\bar{\alpha}(f) \notin P$, so $\bar{\alpha}$ extends to a monomorphism $\alpha: R \rightarrow R$ by defining

$\alpha(fg^{-1}) = \overline{\alpha}(f)\overline{\alpha}(g)^{-1}$, where $f \in S$ and $g \in S - P$.

Now for any $f \in S$, there exists $n \in \mathbb{N}$ such that $\overline{\alpha}^n(f) \notin P$ - therefore, for all $fg^{-1} \in R$, there exists $n \in \mathbb{N}$ with $\alpha^n(fg^{-1})$ a unit. By proposition 1.37, $A(R, \alpha)$ is a field, therefore semi-primitive.

But R is a local ring with unique maximal ideal PR (by [1], chapter 3), which is therefore the Jacobson radical.

Remark:

Note that, in example 3.18, $J(R)$ is not α -invariant. The next result shows that if $J(R)$ is α -invariant, then such examples (i.e. with $A(R, \alpha)$ semiprimitive but R not) do not exist.

3.19 Proposition:

If the Jacobson radical of a ring R is α -invariant, then

- (i) $G(J(R)) \subseteq J(A(R, \alpha))$;
- (ii) If $A(R, \alpha)$ is semiprimitive then so is R .

Proof:

(i) By theorem 2.2, $G(J(R))$ is an ideal of $A(R, \alpha)$, so it will be sufficient to show that $G(J(R))$ is left quasi-regular.

Let $x^{-1}ax^{\dagger} \in G(J(R))$ with $a \in J(R)$. Then, there exists

$c \in R$ with $c(1-a) = 1$, so that $x^{-i}cx^i(1-x^{-i}ax^i) = 1$, and $G(J(R))$ is left quasi-regular.

(ii) Intersecting both sides of (i) with R gives $G(J(R)) \cap R \subseteq J(A(R, \alpha)) \cap R$ or, by theorem 2.2, $\bigcup_{n \geq 0} \alpha^{-n}(J(R)) \subseteq J(A(R, \alpha)) \cap R$. Thus, if $J(A(R, \alpha)) = 0$, then $J(R) = 0$.

From now on, R will be assumed to be left Jordan, but $J(R)$ will not necessarily be α -invariant. Note that the ring R in example 3.18 is left Jordan, but $J(R)$ is not α -invariant.

3.20 Definition:

A closed left ideal M of R is said to be a maximal closed left ideal if it is not strictly contained in any proper closed left ideal.

Remark:

Clearly, if a maximal left ideal happens to be closed, then it is a maximal closed left ideal. Also, if a maximal left ideal M of R is α -invariant, then $R \neq \alpha^{-1}(M) \supseteq M$, so $\alpha^{-1}(M) = M$ and by proposition 2.5, M is closed.

However, example 3.18 shows that even if R is left Jordan, it is possible that none of its maximal ideals are closed.

3.21 Lemma:

Let R be a left Jordan ring, M a left ideal of $A(R, \alpha)$ and denote the α -sequence $\tau(M)$ by $(M_i)_{i \geq 0}$.

Then M is a maximal left ideal iff there exists $k \geq 0$ such that, for each $i \geq k$, M_i is a maximal closed left ideal of R .

Proof:

First, let $k \geq 0$ be such that for all $i \geq k$, M_i is a maximal closed left ideal of R , and assume that there exists a left ideal I of $A(R, \alpha)$ with $M \subsetneq I \subsetneq A(R, \alpha)$.

Then, there exists $i \geq 0$, $r \in R$ such that $x^{-i}rx^i \in I$ but $x^{-i}rx^i \notin M$; i.e. $r \in I_i$ but $r \notin M_i$. For each $j \geq 0$, $\alpha^j(r) \in I_{i+j}$, but if $\alpha^j(r) \in M_{i+j}$, then $r \in M_i$ since $(M_j)_{j \geq 0}$ is an α -sequence. This is impossible, so $\alpha^j(r) \in I_{i+j}$ but $\alpha^j(r) \notin M_{i+j}$ for all $j \geq 0$.

By theorem 1.33, $M_{i+j} \subsetneq I_{i+j} \subsetneq R$ for all $j \geq 0$, which contradicts the definition of k . Thus, M is a maximal left ideal.

Conversely, assume that M is a maximal left ideal of $A(R, \alpha)$ and, if possible, that for each $N \in \mathbb{N}$ there exists $k \geq N$ such that M_k is not a maximal closed left ideal of R .

Since R is left Jordan, there exists $N_1 \in \mathbb{N}$ such that, for all $j \geq 0$, $M_{N_1+j} = \rho_j(M_{N_1})$ and by assumption, there exists $p \geq N_1$ with M_p not a maximal closed left ideal. Thus, there exists a closed left ideal I_p with $R \supsetneq I_p \supsetneq M_p$.

By the proof of theorem 2.6, $\alpha^{-j}(I_p)$ is a closed left ideal for all $j = 1, \dots, p$, and since $(M_i)_{i \geq 0}$ is an α -sequence, $M_j \subseteq \alpha^{-(p-j)}(I_p)$ for $j = 0, \dots, p$.

Now since $M_p \subsetneq I_p$, by proposition 1.35 $\rho_j(M_p) \subsetneq \rho_j(I_p)$ for all $j \geq 0$. Thus the following α -sequence is obtained:

$$I_i = \begin{cases} \alpha^{-(p-i)}(I_p) & \text{for } 0 \leq i \leq p \\ \rho_{i-p}(I_p) & \text{for } i > p \end{cases}$$

with the property that $(I_i)_{i \geq 0} \not\geq (M_i)_{i \geq 0}$.

Applying Δ shows that M cannot be a maximal left ideal. Thus, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, M_k is a maximal closed left ideal.

3.22 Lemma:

Let $\{(I_{k,i})_{i \geq 0} \mid k \in X\}$ be a collection of α -sequences of closed left ideals of R , where X is an indexing set. Then

- (i) $(\bigcap_{k \in X} I_{k,i})_{i \geq 0}$ is an α -sequence of closed left ideals of R ;
- (ii) $\Gamma(\bigcap_{k \in X} \Delta(I_{k,i})_{i \geq 0}) = (\bigcap_{k \in X} I_{k,i})_{i \geq 0}$.

Proof:

(i) Let $i \geq 0$. To show that $\bigcap_{k \in X} I_{k,i}$ is a closed left ideal of R , let $n \geq 0$.

Then $\alpha^{-n}(\text{Ra}^n(\bigcap_{k \in X} I_{k,i})) \subseteq \alpha^{-n}(\text{Ra}^n(I_{k,i}))$ for all $k \in X$

$\subseteq I_{k,i}$ for all $k \in X$, since

each $I_{k,i}$ is closed. Thus $\alpha^{-n}(\text{Ra}^n(\bigcap_{k \in X} I_{k,i})) \subseteq \bigcap_{k \in X} I_{k,i}$.

Now, $\alpha^{-1}(\bigcap_{k \in X} I_{k,i+1}) = \bigcap_{k \in X} \alpha^{-1}(I_{k,i+1})$

$= \bigcap_{k \in X} I_{k,i}$, so that

$(\bigcap_{k \in X} I_{k,i})_{i \geq 0}$ is an α -sequence of closed left ideals of R .

(ii) Fix $j \geq 0$ and let $r \in \Gamma(\bigcap_{k \in X} \Delta(I_{k,i})_{i \geq 0})_j$, i.e.

$x^{-j} r x^j \in \Delta(I_{k,i})_{i \geq 0}$ for all $k \in X$. Since the maps Γ and Δ are mutually inverse (theorem 1.33), this means $r \in I_{k,j}$ for all $k \in X$, and $\Gamma(\bigcap_{k \in X} \Delta(I_{k,i})_{i \geq 0})_j \subseteq \bigcap_{k \in X} I_{k,j}$.

On the other hand, if $r \in \bigcap_{k \in X} I_{k,j}$ then $x^{-j} r x^j \in \bigcap_{k \in X} \Delta(I_{k,i})_{i \geq 0}$, so

$r \in \Gamma(\bigcap_{k \in X} \Delta(I_{k,i})_{i \geq 0})_j$.

Notation:

The collection of all the maximal closed left ideals of R will be denoted by M .

3.23 Theorem:

Let R be a left Jordan ring and denote by $(J_i)_{i \geq 0}$ the α -sequence $\Gamma(J(A(R, \alpha)))$.

Then $J_i = \bigcap_{M \in \mathcal{M}} M$, for all $i \geq 0$.

Proof:

It was shown in lemma 3.21 that a left ideal M of $A(R, \alpha)$ is maximal iff M_i is a maximal closed left ideal, for all i greater than some $k \in \mathbb{N}$.

It is now claimed that, given any $j, k \geq 0$ and any maximal closed left ideal I of R , there exists a maximal left ideal M of $A(R, \alpha)$ such that $M_j = \alpha^{-k}(I)$.

To see this, define an α -sequence $(M_i)_{i \geq 0}$ as follows. Put $M_{j+k} = I$ and $M_i = \alpha^{-(j+k-i)}(I)$ for $i = 0, \dots, j+k$. By the proof of theorem 2.6, each $\alpha^{-(j+k-i)}(I)$ is closed, and $M_j = \alpha^{-k}(I)$.

If $\rho_1(I)$ is not a maximal closed left ideal of R then, since R is left Jordan and therefore has ascending chain condition on closed left ideals, there exists a maximal closed left ideal N which contains $\rho_1(I)$.

By proposition 1.35, $\alpha^{-1}(\rho_1(I)) = I = M_{j+k}$, so $\alpha^{-1}(N) \supseteq I$. But $\alpha^{-1}(N)$ is a closed left ideal, so maximality of I gives $\alpha^{-1}(N) = I$, and the next term in the α -sequence $(M_i)_{i \geq 0}$ may be defined as $M_{j+k+1} = N$.

The procedure can then be repeated for $\rho_1(M_{j+k+1})$, obtaining a maximal closed left ideal M_{j+k+2} such that $\alpha^{-1}(M_{j+k+2}) = M_{j+k+1}$. Continuing the process yields an α -sequence $(M_i)_{i \geq 0}$ of closed left ideals such that $\alpha^{-k}(I) = M_j$, and if $i \geq j+k$ then M_i is a maximal closed left ideal. By lemma 3.21, $M = \Delta((M_i)_{i \geq 0})$ is a maximal left ideal of $A(R, \alpha)$, and the claim is proved.

Now, for any $i \geq 0$,

$$\begin{aligned} J_i &= \left(\bigcap_{\substack{M \text{ a maximal} \\ \text{left ideal} \\ \text{of } A(R, \alpha)}} M \right)_i \\ &= \bigcap_{\substack{M \text{ a maximal} \\ \text{left ideal} \\ \text{of } A(R, \alpha)}} M_i \quad \text{where } \Gamma(M) = (M_i)_{i \geq 0}, \text{ by} \end{aligned}$$

lemma 3.22(ii). But by the claim proved above, for any $k \geq 0$ and $I \in M$, there exists a maximal left ideal M of $A(R, \alpha)$ with $M_i = \alpha^{-k}(I)$. By lemma 3.21, every M_i must be of that form, so

$$J_i = \bigcap_{\substack{k \geq 0 \\ I \in M}} \alpha^{-k}(I). \quad - (1)$$

Now, for each $k \geq 0$, let Λ_k be the collection of all maximal α -sequences $(M_i)_{i \geq 0}$ of closed left ideals of R such that, for all $i \geq k$, M_i is a maximal closed left ideal. By lemma 3.22(i), the

The procedure can then be repeated for $\rho_1(M_{j+k+1})$, obtaining a maximal closed left ideal M_{j+k+2} such that $\alpha^{-1}(M_{j+k+2}) = M_{j+k+1}$. Continuing the process yields an α -sequence $(M_i)_{i \geq 0}$ of closed left ideals such that $\alpha^{-k}(I) = M_j$, and if $i \geq j+k$ then M_i is a maximal closed left ideal. By lemma 3.21, $M = \Delta((M_i)_{i \geq 0})$ is a maximal left ideal of $A(R, \alpha)$, and the claim is proved.

Now, for any $i \geq 0$,

$$\begin{aligned} J_i &= \left(\bigcap_{\substack{M \text{ a maximal} \\ \text{left ideal} \\ \text{of } A(R, \alpha)}} M \right)_i \\ &= \bigcap_{\substack{M \text{ a maximal} \\ \text{left ideal} \\ \text{of } A(R, \alpha)}} M_i \quad \text{where } \Gamma(M) = (M_i)_{i \geq 0}, \text{ by} \end{aligned}$$

lemma 3.22(ii). But by the claim proved above, for any $k \geq 0$ and $I \in M$, there exists a maximal left ideal M of $A(R, \alpha)$ with $M_i = \alpha^{-k}(I)$. By lemma 3.21, every M_i must be of that form, so

$$J_i = \bigcap_{\substack{k \geq 0 \\ I \in M}} \alpha^{-k}(I). \quad - (1)$$

Now, for each $k \geq 0$, let Λ_k be the collection of all maximal α -sequences $(M_i)_{i \geq 0}$ of closed left ideals of R such that, for all $i \geq k$, M_i is a maximal closed left ideal. By lemma 3.22(i), the

sequence $(I_{k,j})_{j \geq 0}$ given by

$$I_{k,j} = \bigcap_{\substack{M_i \\ i \geq 0 \in \Lambda_k}} M_j \quad \text{for each } j \geq 0$$

is an α -sequence of closed left ideals of R : let I_k be the left ideal of $A(R, \alpha)$ so obtained.

Then I_k is α -stable. Indeed, let $r \in I_{k,j}$ for some $j \geq 0$. Let $(M_i)_{i \geq 0} \in \Lambda_k$, and define a new α -sequence $(N_i)_{i \geq 0}$ by putting $N_i = M_{i+1}$ for each $i \geq 0$. Then $(N_i)_{i \geq 0} \in \Lambda_k$ and $N_j = M_{j+1}$. But $r \in I_{k,j}$, so $r \in N_j = M_{j+1}$, whence $r \in I_{k,j+1}$, and $I_{k,j} \subseteq I_{k,j+1}$.

Now, for any $x^{-j} r x^j \in I_k$, i.e. $r \in I_{k,j}$, $\alpha^{-1}(x^{-j} r x^j) = x^{-(j+1)} r x^{j+1} \in I_k$ since $I_{k,j} \subseteq I_{k,j+1}$. Thus $\alpha^{-1}(I_k) \subseteq I_k$, and since R is left Jordan, the ascending chain $I_k \subseteq \alpha(I_k) \subseteq \alpha^2(I_k) \subseteq \dots \subseteq \alpha^n(I_k) \subseteq \dots$ must terminate, giving $\alpha(I_k) = I_k$, and so I_k is α -stable.

By proposition 2.4, $I_{k,k} = I_{k,0}$ for all $k \geq 0$, but

$$I_{k,k} = \bigcap_{M \in M} {}^n M \quad \text{and} \quad I_{k,0} = \bigcap_{\substack{M \in M \\ n \leq k}} {}^n \alpha^{-n}(M).$$

Thus, from (1),

$$J_1 = \bigcap_{M \in M} {}^n M.$$

3.24 Corollary:

If R is a left Jordan ring such that the intersection of all the maximal closed left ideals is zero, then $A(R, \alpha)$ is semiprimitive.

Proof:

By theorem 3.23, $J_i = 0$ for all $i \geq 0$.

3.25 Corollary:

If R is a left and right Jordan ring then the intersection of all the maximal closed left ideals is the same as the intersection of all the maximal closed right ideals.

Proof:

Since the Jacobson radical is left-right symmetric, the right-handed version of the above work shows that each term of $\Gamma(J(A(R, \alpha)))$ is given by the intersection of all the maximal closed right ideals.

3.26 Examples:

(i) Consider the ring R of example 3.18. R is left Jordan, and the only closed left ideal, apart from R itself, is 0 . By corollary 3.24, $A(R, \alpha)$ is semiprimitive, as was seen in example 3.18.

(ii) Let K be a field, $\sigma: K \rightarrow K$ a monomorphism which is not an automorphism. Let $S = K[y]$ and define $\bar{\alpha}: S \rightarrow S$ by

$$\bar{\alpha}\left(\sum_{i=0}^n f_i y^i\right) = \sum_{i=0}^n \sigma(f_i) y^i. \text{ Then } S \text{ is left Jordan, by theorem 3.3.}$$

Now define $R = \begin{bmatrix} S & S \\ 0 & S \end{bmatrix}$ with $\alpha: R \rightarrow R$ defined

by $\alpha \begin{bmatrix} s_1 & s_2 \\ 0 & s_3 \end{bmatrix} = \begin{bmatrix} \bar{\alpha}(s_1) & \bar{\alpha}(s_2) \\ 0 & \bar{\alpha}(s_3) \end{bmatrix}.$

Since the map ψ in the proof of theorem 3.2 restricts to an isomorphism between $A(\bar{M}_n(S), \alpha)$ and $\bar{M}_n(A(S, \bar{\alpha}))$ where \bar{M}_n denotes the upper triangular matrix ring, (R, α) is left Jordan.

The maximal left ideals of R are

$$I_1 = \begin{bmatrix} S & S \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad I_2 = \begin{bmatrix} 0 & S \\ 0 & S \end{bmatrix}, \quad \text{each}$$

of which is α -invariant and therefore closed, by the remark following definition 3.20.

Theorem 3.23 therefore gives $J(A(R, \alpha))_l = I_1 \cap I_2$

$$= \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}.$$

§5. Maximal Left Ideals of Left Artinian Rings.

This section has two objectives, the first of which is to show that maximal left ideals of left Artinian rings are closed. If R is a left Artinian ring with Jacobson radical $J(R)$ then, as was seen in section 3 of chapter 1, the semiprimitive idempotents of $R/J(R)$ are defined to be those central idempotents of $R/J(R)$ which generate minimal ideals. It was shown in theorem 1.15 that if $\{f_i \mid i = 1, \dots, n\}$ are orthogonal

idempotents of R such that $\{\phi(f_i) \mid i = 1, \dots, n\}$ are precisely the semiprimitive idempotents of $R/J(R)$ (ϕ being the natural surjection), then $f_i R f_i$ is a primary ring, for each $i = 1, \dots, n$. By theorem 1.10, a primary ring is a matrix ring over a completely primary ring, and the three-step method of proof described in section 3 of chapter 1 will be used here. Thus, maximal left ideals will first be shown to be closed in completely primary rings, then in primary rings, and finally in left Artinian rings. Note that the phrase "each maximal left ideal is closed" is taken to mean that the maximal left ideals are closed under any monomorphism $\alpha: R \rightarrow R$. Also, note that if M is a maximal left ideal which is not closed, then for some $k \geq 0$, $M \not\subseteq \alpha^{-k}(R\alpha^k(M))$, so that $1 \in \alpha^{-k}(R\alpha^k(M))$, and since $\alpha(1) = 1$, $1 \in R\alpha^k(M)$. This fact will be used frequently.

The second aim of the section is to use theorem 3.23, which gives the α -sequence of $J(A(R, \alpha))$, to generalize a result of Jategaonkar [13] which states that if R is left Artinian then $\alpha^{-1}(J(R)) = J(R)$.

Specifically, the generalization is that the left Artinian condition on R may be replaced by the assumption that R is left Jordan, and that maximal left ideals of R are closed. Observe that by the first part of the section, a left Artinian ring satisfies these conditions. An example is provided to show that this is a genuine generalization.

In this section, ϕ will denote the natural surjection from R onto $R/J(R)$, and T will denote the identity element of $R/J(R)$.

3.27 Proposition:

If R is a completely primary ring and M is the unique maximal left ideal of R , then M is closed.

Proof:

Since $M = J(R)$ and R is left Artinian, M is nilpotent. Therefore $\alpha^n(M)$ is a nil subring for any $n \geq 0$, so $\alpha^n(M) \subseteq M$. Hence $R\alpha^n(M) \subseteq M$, and since $\alpha^{-n}(M)$ is a nilpotent left ideal, $\alpha^{-n}(R\alpha^n(M)) \subseteq M$, and M is closed.

Notation:

Let S be a completely primary ring, and let $i, n \in \mathbb{N}$. Then K_i will denote the maximal left ideal of $M_n(S)$ obtained by insisting that the entries in the i^{th} column are elements of $J(S)$, the other entries being arbitrary. K_i will be called the i^{th} standard maximal left ideal of $M_n(S)$.

3.28 Proposition:

Let S be a completely primary ring, and let M be a maximal left ideal of $M_n(S)$. Then there exists $i \in \mathbb{N}$ and a unit u of $M_n(S)$ such that $M = K_i u$.

Proof:

First, consider the case where S is a division ring, and let M be a maximal left ideal of the simple Artinian ring $M_n(S)$. By

theorem 1.12 of [6], $M = M_n(S)e$ for some idempotent e of $M_n(S)$. The idempotent element $1 - e$ is primitive, since the fact that $M_n(S) = M_n(S)e \oplus M_n(S)(1-e)$ means that $M_n(S)(1-e) \cong \frac{M_n(S)}{M_n(S)e}$, and therefore that $M_n(S)(1-e)$ is a minimal left ideal.

By lemma 1.10 of [6], $e = e_2 + e_3 + \dots + e_k$ where the e_i are mutually orthogonal, primitive idempotents, so with $e_1 = 1 - e$, $1 = e_1 + e_2 + \dots + e_k$ and again the e_i are mutually orthogonal.

Now let $\{f_i \mid i = 1, \dots, n\}$ be the standard primitive orthogonal idempotents for $M_n(S)$ (i.e. f_i is the matrix with 1 in the (i,i) -entry and zero elsewhere). By lemmas 1.13 and 1.14, there exists a unit u of $M_n(S)$ such that, reordering the f_i if necessary, $e_i = u^{-1}f_i u$ for all $i = 1, \dots, n$, and $k = n$.

$$\begin{aligned} \text{Then,} \quad M &= M_n(S)e_2 \oplus \dots \oplus M_n(S)e_k \\ &= (M_n(S)f_2 \oplus \dots \oplus M_n(S)f_k)u \\ &= K_i u \quad \text{for some } 1 \leq i \leq n. \end{aligned}$$

Now consider the case where S is completely primary, and let M be a maximal left ideal of $M_n(S)$. Since (by theorem 3, p.11 of [11]) $J(M_n(S)) = M_n(J(S))$, there is an isomorphism

$$\begin{aligned} \psi: \frac{M_n(S)}{J(M_n(S))} &\rightarrow M_n(S/J(S)). \text{ Let } \phi \text{ denote the natural surjection from} \\ M_n(S) &\text{ to } \frac{M_n(S)}{J(M_n(S))} \text{ and let } \lambda \text{ denote the map } \psi \circ \phi: M_n(S) \rightarrow M_n(S/J(S)). \end{aligned}$$

Then $\lambda(M)$ is a maximal left ideal of $M_n(S/J(S))$. Since $S/J(S)$ is a division ring, $\lambda(M) = \overline{K_i} \overline{u}$ where $\overline{K_i}$ is the i^{th} standard maximal left ideal of $M_n(S/J(S))$, and \overline{u} is a unit of $M_n(S/J(S))$. But since units can be lifted over nil ideals, there exists a unit u of $M_n(S)$ such that $\lambda(u) = \overline{u}$, and clearly $\lambda^{-1}(\overline{K_i}) = K_i$, where K_i is the i^{th} standard maximal left ideal of $M_n(S)$.

Therefore, $\lambda(K_i u) = \overline{K_i} \overline{u}$ and

$$M \subseteq \lambda^{-1}(\lambda(M)) = \lambda^{-1}(\overline{K_i} \overline{u}) \supseteq K_i u.$$

By maximality of M and $K_i u$, this implies $M = K_i u$.

3.29 Lemma:

Let $u \in R$ be a unit, M any subset of R , and assume that $1 \in R\alpha^n(M)$ for some $n \geq 1$. Then $1 \in R(\tilde{u}_0\alpha)^n(M)$ where $\tilde{u}: R \rightarrow R$ is defined by $\tilde{u}(r) = u^{-1}ru$.

Proof:

The proof is by induction on n . Assume that $1 \in R\alpha(M)$. Then $1 = \sum_i r_i \alpha(m_i)$ where $r_i \in R$ and $m_i \in M$, and $1 = \sum_i (u^{-1}r_i u)(u^{-1}\alpha(m_i)u) \in R\tilde{u}_0\alpha(M)$, so the assertion holds for $n = 1$.

Now assume the conclusion to be true for $n-1$, and assume $1 \in R\alpha^n(M)$, i.e. $1 = \sum_i r_i \alpha^n(m_i)$ for $r_i \in R$, $m_i \in M$. Then $\alpha^{n-1}(u) = \sum_i r_i \alpha^n(m_i) \alpha^{n-1}(u)$, and since $\alpha^{n-1}(u)$ is a unit, with inverse $\alpha^{n-1}(u^{-1})$,

$$1 = \sum_i \alpha^{n-1}(u)^{-1} r_i \alpha^{n-1}(u) \alpha^{n-1}(u^{-1}) \alpha^{n-1}(m_i) \alpha^{n-1}(u) .$$

$$\text{So } 1 = \sum_i \alpha^{n-1}(u)^{-1} r_i \alpha^{n-1}(u) \alpha^{n-1}(u^{-1}) \alpha^{n-1}(m_i) u \in R \alpha^{n-1}(\tilde{u}_0^\alpha(M)) .$$

By the induction hypothesis, $1 \in R(\tilde{u}_0^\alpha)^{n-1}(\tilde{u}_0^\alpha(M))$,

i.e. $1 \in R(\tilde{u}_0^\alpha)^n(M)$, and the result is proved.

3.30 Lemma:

Let R be a left Artinian ring with Jacobson radical $J(R)$, and let $\{\bar{e}_i \mid i = 1, \dots, n\}$ be a set of mutually orthogonal primitive idempotents of $R/J(R)$ such that $\sum_{i=1}^n \bar{e}_i = \bar{1}$.

If $\{g_i \mid i = 1, \dots, n\}$ is any set of mutually orthogonal, non-zero idempotents of R with $\sum_{i=1}^n g_i = 1$, then each g_i is primitive.

Proof:

Assume that, for some $1 \leq k \leq n$, g_k is not primitive.

Then $g_k = g_k' = g_k' + g_k''$ where g_k' and g_k'' are non-zero, orthogonal idempotents.

Since $g_k' = g_k' g_k$ and $g_k'' = g_k'' g_k$, for any $1 \leq i \leq n$ with $i \neq k$, $g_k' g_i = (g_k' g_k) g_i = 0$, and similarly $g_k'' g_i = 0$.

Thus $\{g_1, \dots, g_{k-1}, g_k', g_k'', g_{k+1}, \dots, g_n\}$ is a set of $n+1$ non-zero mutually orthogonal idempotents.

Since R is a ring with unity, $J(R)$ cannot contain any idempotents, so the set $\{\phi(g_1), \dots, \phi(g_{k-1}), \phi(g'_k), \phi(g''_k), \phi(g_{k+1}), \dots, \phi(g_n)\}$ consists of $n+1$ mutually orthogonal, non-zero idempotents of $R/J(R)$, where $\phi: R \rightarrow R/J(R)$ denotes the natural surjection.

But this means that $R/J(R)$ can be written as a direct sum of $n+1$ non-zero left ideals, and since $R/J(R) = \bigoplus_{i=1}^n R/J(R)\bar{e}_i$ (each $R/J(R)\bar{e}_i$ being minimal), this contradicts the Krull-Schmidt-Azumaya theorem ([4], theorem 6.12).

3.31 Theorem:

Every maximal left ideal of a primary ring is closed.

Proof:

Let R and S be isomorphic rings, $\psi: R \rightarrow S$ an isomorphism, and let M be a maximal left ideal which is not closed under the monomorphism $\alpha: R \rightarrow R$.

Then $\psi(M)$ is not closed under the endomorphism $\psi \alpha \psi^{-1}$ of S . Therefore, by theorem 1.10, it is enough to prove the result for rings of the form $M_n(S)$ where S is completely primary.

Now if M is a maximal left ideal of $M_n(S)$, then by proposition 3.28, $M = K_i u$ for some $1 \leq i \leq n$ and some unit u of $M_n(S)$. If M is not closed, then for some $k \geq 1$, $1 \in M_n(S) \alpha^k(M)$, or $1 \in M_n(S) \alpha^k(K_i u)$.

Thus, $1 \in M_n(S)\alpha^k(K_i)\alpha^k(u)$, which means that $1 \in M_n(S)\alpha^k(K_i)$, so K_i is not closed either.

It is therefore sufficient to show that the standard maximal left ideals of $M_n(S)$ are closed.

Let $\{e_{ij} \mid i, j = 1, \dots, n\}$ be the standard set of matrix units for $M_n(S)$ - i.e. e_{ij} is the matrix whose (i, j) -entry is 1, but has all other entries zero. Since, for each $i = 1, \dots, n$, $S = e_{ii}M_n(S)e_{ii}$, all the rings $e_{ii}M_n(S)e_{ii}$ are completely primary.

But $\{\alpha(e_{ij}) \mid i, j = 1, \dots, n\}$ is another set of matrix units for $M_n(S)$, so the set $\{\alpha(e_{ii}) \mid i = 1, \dots, n\}$ consists of mutually orthogonal non-zero idempotents, such that $\sum_{i=1}^n \alpha(e_{ii}) = 1$.

Since $\frac{M_n(S)}{J(M_n(S))} \cong M_n(S/J(S))$ (as in the proof of proposition 3.28),

and since $S/J(S)$ is a division ring, $\frac{M_n(S)}{J(M_n(S))}$ is a simple Artinian

ring and there exist primitive, mutually orthogonal idempotents

$\{\bar{g}_i \mid i = 1, \dots, n\}$ of $\frac{M_n(S)}{J(M_n(S))}$ with $\sum_{i=1}^n \bar{g}_i = \bar{1}$. By lemma 3.30,

$\alpha(e_{ii})$ is primitive, for each $i = 1, \dots, n$.

Therefore, by lemma 1.14, $\alpha(e_{ii})M_n(S)\alpha(e_{ii})$ is completely primary for each $i = 1, \dots, n$, and by lemma 1.12, there exists a unit u of $M_n(S)$ such that $e_{ij} = \tilde{u}_0 \alpha(e_{ij})$, for each $i, j = 1, \dots, n$.

Now, for each $i = 1, \dots, n$, define a map $\psi_i: S \rightarrow S$ by $\psi_i(s) = (\tilde{u}_0^\alpha(se_{ii}))_{ii}$ where se_{ii} is the matrix with s in the (i,i) -place and zeros elsewhere, and $(m)_{ii}$ denotes the (i,i) -entry of the matrix m . Then ψ_i is a ring homomorphism. Indeed, for $s, t \in S$,

$$\begin{aligned}\psi_i(s+t) &= (\tilde{u}_0^\alpha((s+t)e_{ii}))_{ii} \\ &= (\tilde{u}_0^\alpha(se_{ii} + te_{ii}))_{ii} \\ &= (\tilde{u}_0^\alpha(se_{ii}) + \tilde{u}_0^\alpha(te_{ii}))_{ii} \text{ since } \tilde{u}_0^\alpha \text{ is additive,} \\ &= (\tilde{u}_0^\alpha(se_{ii}))_{ii} + (\tilde{u}_0^\alpha(te_{ii}))_{ii}, \text{ so } \psi_i\end{aligned}$$

is additive. Also,

$$\begin{aligned}\psi_i(st) &= (\tilde{u}_0^\alpha((st)e_{ii}))_{ii} \\ &= (\tilde{u}_0^\alpha(se_{ii}te_{ii}))_{ii} \\ &= (\tilde{u}_0^\alpha(se_{ii})\tilde{u}_0^\alpha(te_{ii}))_{ii} \text{ since } \tilde{u}_0^\alpha \text{ is a ring}\end{aligned}$$

homomorphism. But since $\tilde{u}_0^\alpha(e_{ii}) = e_{ii}$,

$\tilde{u}_0^\alpha(se_{ii}) = \tilde{u}_0^\alpha(e_{ii}se_{ii}^2) = e_{ii}\tilde{u}_0^\alpha(se_{ii})e_{ii}$, so that $\tilde{u}_0^\alpha(se_{ii})$ is a matrix with zeros everywhere except possibly the (i,i) -position, for any $s \in S$.

$$\text{Thus } (\tilde{u}_0^\alpha(se_{ii})\tilde{u}_0^\alpha(te_{ii}))_{ii} = (\tilde{u}_0^\alpha(se_{ii}))_{ii}(\tilde{u}_0^\alpha(te_{ii}))_{ii}$$

and ψ_i is a ring homomorphism.

It is injective because if $\psi_i(s) = 0$, then since every entry of the matrix $\tilde{u}_0^\alpha(se_{ii})$ except possibly the (i,i) -entry is zero anyway, $\tilde{u}_0^\alpha(se_{ii}) = 0$. Since \tilde{u}_0^α is a monomorphism, $se_{ii} = 0$, hence $s = 0$, and ψ_i is injective.

Let M be the j^{th} standard maximal left ideal of $M_n(S)$ and, if possible, assume that $\tilde{u}_0^\alpha(M) \not\subseteq M$. Then for some $m \in M$, $\tilde{u}_0^\alpha(m)$ has a unit of S appearing in the j^{th} column - say $(\tilde{u}_0^\alpha(m))_{pj}$ is a unit of S . But $e_{jp}m \in M$, and $\tilde{u}_0^\alpha(e_{jp}m) = e_{jp}\tilde{u}_0^\alpha(m)$ (since $\tilde{u}_0^\alpha(e_{ij}) = e_{ij}$ for all $i,j = 1,\dots,n$), and $e_{jp}\tilde{u}_0^\alpha(m)$ has a unit of S appearing in its (j,j) -place.

Therefore, in order to show that $\tilde{u}_0^\alpha(M) \subseteq M$, it is sufficient to show that for any $m \in M$, $(\tilde{u}_0^\alpha(m))_{jj}$ cannot be a unit of S .

Now, for any $m \in M$,

$$\begin{aligned} (\tilde{u}_0^\alpha(m))_{jj} &= (e_{jj}\tilde{u}_0^\alpha(m)e_{jj})_{jj} \\ &= (\tilde{u}_0^\alpha(e_{jj}me_{jj}))_{jj} \\ &= \psi_j(m_{jj}). \end{aligned}$$

Since ψ_j is a monomorphism of the completely primary ring S , and $m_{jj} \in J(S)$, proposition 3.27 shows that $\psi_j(m_{jj}) \in J(S)$, and is

therefore never a unit of S .

Thus, $\tilde{u}_0^\alpha(M) \subseteq M$, and for any $k \geq 0$, $1 \notin M_n(S)(\tilde{u}_0^\alpha)^k(M)$. By lemma 3.29, $1 \notin M_n(S)\alpha^k(M)$ for any $k \geq 0$, and by the comments at the beginning of the proof, every maximal left ideal of a primary ring is closed.

3.32 Lemma:

Let R be a semisimple Artinian ring, M a maximal left ideal of R , and $e \in R$ an idempotent. Then either $eMe = eRe$ or eMe is a maximal left ideal of eRe .

Proof:

Assume that $eMe \neq eRe$.

Since R is semisimple Artinian, there exists a left ideal K of R such that $R = M \oplus K$ (by lemma 1.9 of [6]).

$$\text{Therefore } eRe = eMe + eKe \quad - \quad (1).$$

It is claimed that eKe is a minimal left ideal of eRe . Indeed, let X be a non-zero left ideal of eRe with $X \subseteq eKe$. Then RX is a left ideal of R , and $0 \neq RX \subseteq Ke$.

Since K is a minimal left ideal, the map $\psi: K \rightarrow Ke$ given by $\psi(k) = ke$ has kernel either K or 0 , and if $\ker \psi = K$ then $Ke = 0$, which implies from (1) that $eMe = eRe$. This contradicts the earlier assumption that $eRe \neq eMe$, therefore $\ker \psi = 0$ and Ke is a minimal left ideal of R .

Thus, $RX = Ke$ and since $X \subseteq eKe$, $eReX = eKe$. Therefore $X = eKe$ and eKe is a minimal left ideal.

Now this means that either $eKe \cap eMe = 0$ or $eKe \cap eMe = eKe$. The second alternative is impossible because it would mean $eKe \subseteq eMe$ and, from (1), that $eRe = eMe$. Therefore $eKe \cap eMe = 0$, and the sum at (1) is direct, showing that eMe is a maximal left ideal of eRe .

3.33 Theorem:

Every maximal left ideal of a left Artinian ring is closed.

Proof:

Let R be a left Artinian ring, let M be a maximal left ideal of R , and let $\{\bar{f}_i \mid i = 1, \dots, n\}$ be the semiprimitive idempotents of $R/J(R)$. By lemma 1.10 of [6], each \bar{f}_i may be written as the sum of mutually orthogonal primitive idempotents, and these may be numbered such that, for $0 = k_0 < k_1 < \dots < k_n = m$,

$$\bar{f}_i = \sum_{j=k_{i-1}+1}^{k_i} \bar{e}_j \quad (1)$$

where each \bar{e}_j is a primitive idempotent, for each $i = 1, \dots, n$.

Since $R/J(R)$ (which will be denoted \bar{R}) is the direct sum of the ideals generated by the \bar{f}_i , $\bar{1} = \sum_{i=1}^n \bar{f}_i \bar{r}_i$ for $\bar{r}_i \in \bar{R}$. Since the \bar{f}_i are orthogonal, this gives $\bar{f}_i = \bar{f}_i \bar{r}_i$, whence $\sum_{i=1}^n \bar{f}_i = \bar{1}$, and $\sum_{j=1}^m \bar{e}_j = \bar{1}$.

By proposition 5, p.54 of [11] then, there exist orthogonal idempotents $\{e_i \mid i = 1, \dots, m\}$ of R such that $\phi(e_i) = \bar{e}_i$ for each $i = 1, \dots, m$, and $\sum_{i=1}^m e_i = 1$. By lemma 3.30, each e_i must be primitive.

Putting $f_i = \sum_{j=k_{i-1}+1}^{k_i} e_j$ for $i = 1, \dots, n$ gives a set

$\{f_i \mid i = 1, \dots, n\}$ of orthogonal idempotents of R such that

$$\sum_{i=1}^n f_i = 1 \text{ and } \phi(f_i) = \bar{f}_i, \text{ for } i = 1, \dots, n.$$

It is now claimed that, for some $1 \leq i \leq n$, $f_i M f_i$ is a maximal left ideal of the subring $f_i R f_i$ of R .

Indeed, there exists $1 \leq i \leq n$ such that $\bar{f}_i \bar{M} \bar{f}_i$ is a maximal left ideal of $\bar{f}_i \bar{R} \bar{f}_i$, otherwise, by lemma 3.32, $\bar{f}_j \bar{M} \bar{f}_j = \bar{f}_j \bar{R} \bar{f}_j$ for each $j = 1, \dots, n$, so that $\bar{f}_j \in \bar{f}_j \bar{M} \bar{f}_j$ for each $j = 1, \dots, n$. Since each \bar{f}_j is central, and since \bar{M} is a left ideal of \bar{R} , this means that $\bar{f}_j \in \bar{M}$ for all $1 \leq j \leq n$, and therefore that $\bar{1} \in \bar{M}$.

The natural surjection $\phi: R \rightarrow R/J(R)$, when restricted to $f_i R f_i$, has image $\bar{f}_i \bar{R} \bar{f}_i$ and kernel $f_i R f_i \cap J(R) = J(f_i R f_i)$ (by theorem 1.15).

Thus $\frac{f_i R f_i}{J(f_i R f_i)} \cong \bar{f}_i \bar{R} \bar{f}_i$. By theorem 1.15, $J(f_i R f_i) = f_i J(R) f_i \subseteq f_i M f_i$,

and the image of $\frac{f_i M f_i}{J(f_i R f_i)}$ is $\bar{f}_i \bar{M} \bar{f}_i$, which by the preceding paragraph

is a maximal left ideal of $\overline{f_i R} f_i$. Therefore $\frac{f_i M f_i}{J(f_i R f_i)}$ is a maximal left ideal of $\frac{f_i R f_i}{J(f_i R f_i)}$.

Hence, $f_i M f_i$ is a maximal left ideal of $f_i R f_i$ and the claim is proved.

Now, it is clear that $\{\alpha(e_i) \mid i = 1, \dots, m\}$ is a set of mutually orthogonal idempotents of R with $\sum_{i=1}^m \alpha(e_i) = 1$, so by lemma 3.30, each $\alpha(e_i)$ is primitive.

By lemmas 1.13 and 1.14 therefore, there exists a unit u of R and a permutation π on $\{1, \dots, m\}$ such that $\tilde{u}_0 \alpha(e_i) = e_{\pi(i)}$, for each $i = 1, \dots, m$.

If p denotes the period of π , then $(\tilde{u}_0 \alpha)^p(e_i) = e_i$ for each $i = 1, \dots, m$.

Now assume M is not closed, so that for some $k \geq 0$, $1 \in Ra^k(M)$.

By lemma 3.29, $1 \in R(\tilde{u}_0 \alpha)^k(M)$ - (2).

Let q be the smallest integer such that $pq \geq k$.

Applying $(\tilde{u}_0 \alpha)^{pq-k}$ to both sides of (2) gives

$$1 \in R(\tilde{u}_0 \alpha)^{pq}(M) \quad - \quad (3).$$

The monomorphism $(\tilde{u}_0 \alpha)^p$ will be denoted by β , so, multiplying both sides of (3) on the right by f_i gives $f_i \in R\beta^q(Mf_i)$, since

$\beta(f_i) = f_i$. Therefore,

$$f_i = \sum_j r_j \beta^Q(m_j f_i) \quad \text{where } r_j \in R, m_j \in M.$$

Multiplying on the left by f_i gives

$$f_i = \sum_j f_i r_j \beta^Q(m_j f_i) = \sum_j f_i^2 r_j \beta^Q(m_j f_i) .$$

Since \bar{f}_i is central in $R/J(R)$, $f_i r_j = r_j f_i + a_j$ where $a_j \in J(R)$, for each j , so

$$\begin{aligned} f_i &= \sum_j f_i (r_j f_i + a_j) \beta^Q(m_j f_i) \quad \text{where } a_j \in J(R) \\ &= \sum_j (f_i r_j f_i) f_i \beta^Q(m_j f_i) + \sum_j f_i a_j \beta^Q(m_j f_i) \\ &= \sum_j (f_i r_j f_i) \beta^Q(f_i m_j f_i) + \sum_j f_i a_j \beta^Q(m_j) f_i , \text{ since} \end{aligned}$$

$\beta(f_i) = f_i$. Since $a_j \in J(R)$, the second summand is an element of $f_i J(R) f_i = J(f_i R f_i)$ (by theorem 1.15) and since f_i is the unity of $f_i R f_i$, $f_i - \sum_j f_i a_j \beta^Q(m_j) f_i = \sum_j (f_i r_j f_i) \beta^Q(f_i m_j f_i)$ is a unit of $f_i R f_i$.

Therefore, the maximal left ideal $f_i M f_i$ of the primary ring $f_i R f_i$ is not closed under the monomorphism β . This contradicts theorem 3.31, so M must be closed.

3.34 Theorem:

Let (R, α) be left Jordan, such that each maximal left ideal of R is closed under α . Then, $\alpha^{-1}(J(R)) = J(R)$.

Proof:

Let $(J_i)_{i \geq 0}$ denote the α -sequence of the Jacobson radical of $A(R, \alpha)$. Then by theorem 3.23, $J_i = \bigcap_{M \in M} M$ where M is the collection of all maximal closed left ideals of R . But since each maximal left ideal of R is closed, M consists precisely of the maximal left ideals of R , and therefore $J_i = J(R)$ for all $i \geq 0$. Since $(J_i)_{i \geq 0}$ is an α -sequence, $\alpha^{-1}(J(R)) = J(R)$, and the result is proved.

3.35 Remark:

In lemma 1.1 of [13], Jategaonkar proves that if R is a left Artinian ring and $\alpha: R \rightarrow R$ is a monomorphism, then $\alpha^{-1}(J(R)) = J(R)$. By proposition 3.1, a left Artinian ring is left Jordan, and by theorem 3.33, all the maximal left ideals of R are closed. So theorem 3.34 contains Jategaonkar's result as a special case.

To see that theorem 3.34 is a genuine generalization of the left Artinian case, it is only necessary to note that there exist rings R and monomorphisms $\alpha: R \rightarrow R$ such that (R, α) is left Jordan, each maximal left ideal is closed under α , but R is not left Artinian. The following example gives such a ring.

3.36 Example:

Let K be a field with a monomorphism $\sigma: K \rightarrow K$ which is not surjective, let $S = K[y]$ where y is an indeterminate, and define $\bar{\alpha}: S \rightarrow S$ by $\bar{\alpha}(\sum_i f_i y^i) = \sum_i \sigma(f_i) y^i$. By theorem 3.3, $(S, \bar{\alpha})$ is left Jordan. Let P denote the prime ideal of S generated by y . If $f \in S$ is such that $f \notin P$ then $\bar{\alpha}(f) \notin P$, so $\bar{\alpha}$ extends to a monomorphism $\alpha: R \rightarrow R$ where R is the localization of S at P . By theorem 3.4, (R, α) is left Jordan.

By chapter 3 of [1], R is a local ring with unique maximal ideal PR . If $f \in PR$ then $f = \sum_i f_i (g_i h_i^{-1})$ where $f_i \in P$, $h_i \in P'$ and $g_i \in S$. Therefore, $\alpha(f) = \sum_i \bar{\alpha}(f_i) \bar{\alpha}(g_i) \bar{\alpha}(h_i)^{-1} \in PR$ since $\bar{\alpha}(P) \subseteq P$; so PR is α -invariant. By the remark following definition 3.20, PR is closed.

However, R is not Artinian since it has an infinite descending chain $(y^n R)_{n=1}^{\infty}$ of ideals.

CHAPTER 4.

THE QUOTIENT RING PROBLEM FOR $A(R, \alpha)$.

This chapter is concerned with two aspects of the quotient ring problem for $A(R, \alpha)$. The first part of the chapter deals with conditions on R equivalent to $A(R, \alpha)$ being a full quotient ring, the second part with the question of when $A(R, \alpha)$ has a left Artinian left quotient ring.

It is straightforward to obtain a characterization in terms of elements of when $A(R, \alpha)$ is a quotient ring, but the main object here will be to obtain an element-free condition. This is done under the assumption that R is left and right Jordan, and uses a theorem (13.10 of [3]), of Stafford, which states that a Noetherian ring S is a full quotient ring if and only if $r(A) \cap \ell(A) \subseteq J$ where A denotes the Artinian radical and J denotes the Jacobson radical of S . The assumption that R is left and right Jordan enables the α -sequences which correspond to Artinian left ideals of $A(R, \alpha)$ to be determined, and this is combined with the work on the Jacobson radical of $A(R, \alpha)$ in the previous chapter to yield a Stafford-like criterion on R .

Recall that an Artinian left ideal of a ring R is a left ideal which is Artinian as a left R -module, and also that a module has finite length iff it is both Artinian and Noetherian.

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§1. $A(R, \alpha)$ as a Quotient Ring.

4.1 Proposition:

If R is a quotient ring, then $A(R, \alpha)$ is also a quotient ring.

Proof:

Let $x^{-i}rx^i$ be a regular element of $A(R, \alpha)$. Then by proposition 1.25, $\alpha^n(r)$ is a regular element of R , for each $n \geq 0$, and therefore r is a unit of R .

Hence, $x^{-i}rx^i$ is a unit of $A(R, \alpha)$.

Remark:

The converse clearly does not hold. Indeed, let $S = K[x_i]_{i \in \mathbb{Z}}$ where K is a field and the x_i are indeterminates, and let $\alpha: S \rightarrow S$ be the K -endomorphism such that $\alpha(x_i) = x_{i+1}$. Let R be S localized at the set $K[x_i]_{i \geq 0}$ and extend α from S to R in the obvious manner. Then R is not a quotient ring, but $A(R, \alpha)$ is a field, by proposition 1.27.

It is easy to obtain an element-wise condition on R which is equivalent to $A(R, \alpha)$ being a quotient ring:

4.2 Proposition:

$A(R, \alpha)$ is a quotient ring iff for any $r \in R$ such that $\alpha^n(r)$ is regular for all $n \geq 0$, there exists $m \geq 0$ such that $\alpha^m(r)$ is a unit of R .

Proof:

If $A(R, \alpha)$ is a quotient ring, then all its regular elements are units. Thus, if $\alpha^n(r)$ is a regular element of R for all $n \geq 0$, then $x^{-i}rx^i$ is a regular element of $A(R, \alpha)$, by proposition 1.25, for each $i \geq 0$. Therefore $x^{-i}rx^i$ is a unit of $A(R, \alpha)$, or by 1.27, $\alpha^m(r)$ is a unit for some $m \geq 0$.

Conversely, if $x^{-i}rx^i \in A(R, \alpha)$ is regular, then $\alpha^n(r)$ is regular in R for all $n \geq 0$, so there exists $m \geq 0$ with $\alpha^m(r)$ a unit. Thus, $x^{-i}rx^i$ is a unit, and $A(R, \alpha)$ is a quotient ring.

The aim here, however, is to obtain an element-free characterization of when $A(R, \alpha)$ is a quotient ring.

4.3 Definition:

An α -sequence $(I_i)_{i \geq 0}$ of closed left ideals is said to be bounded if $\sup\{s(i) \mid i \geq 0\} < \infty$, where $s(i)$ denotes the supremum of the lengths of chains of closed left ideals contained in I_i .

4.4 Lemma:

Let I be a left ideal of $A(R, \alpha)$. Then

- (i) I has finite length (as a left $A(R, \alpha)$ -module) iff $(I_i)_{i \geq 0}$ is a bounded α -sequence;
- (ii) In this case, $(I_i)_{i \geq 0}$ is a stable α -sequence.

Proof:

First assume that I has finite length, say n , as a left

$A(R, \alpha)$ -module. Suppose that, for some $k \geq 0$, I_k contains a chain of closed left ideals of length greater than n , i.e.

$$I_k = J_0 \supsetneq J_1 \supsetneq \dots \supsetneq J_{n+1} = 0.$$

Then, for any $i \geq 0$, by propositions 1.35 and 1.36,

$$I_{k+i} \supseteq \rho_i(I_k) = \rho_i(J_0) \supsetneq \rho_i(J_1) \supsetneq \dots \supsetneq \rho_i(J_{n+1}) = 0.$$

Now, for $j = 0, \dots, n+1$, let $(K_{ji})_{i \geq 0}$ be the α -sequence of closed left ideals defined by

$$K_{ji} = \begin{cases} \alpha^{-(k-i)}(J_j) & 0 \leq i \leq k \\ \rho_{i-k}(J_j) & i \geq k. \end{cases}$$

If the α -sequence $(K_{ji})_{i \geq 0}$ is denoted just as K_j , then

$$(I_i)_{i \geq 0} \supseteq K_0 > K_1 > \dots > K_{n+1} = 0,$$

and applying Δ gives a chain

$$I \supseteq \Delta(K_0) \supsetneq \Delta(K_1) \supsetneq \dots \supsetneq \Delta(K_{n+1}) = 0$$

of length at least $n+1$, consisting of distinct left $A(R, \alpha)$ -submodules of I .

This contradicts the fact that I has length n .

Before proving the converse of (i), it is convenient to prove (ii), namely that a bounded α -sequence of closed left ideals is stable.

Let $(I_i)_{i \geq 0}$ be a bounded α -sequence of closed left ideals, and let $n = \sup\{s(i) \mid i \geq 0\}$. Since $(s(i))_{i \geq 0}$ is a sequence of non-negative integers, it must attain its supremum; therefore there exists $N \geq 0$ such that I_N contains a chain of n closed left ideals, i.e.

$$I_N = L_0 \supsetneq L_1 \supsetneq \dots \supsetneq L_n = 0.$$

Let $k \geq 0$ and suppose that $I_{N+k} \neq \rho_k(I_N)$. By proposition 1.36, this means that $\rho_k(I_N) \subsetneq I_{N+k}$ so that I_{N+k} contains the chain

$$I_{N+k} \supsetneq \rho_k(L_0) \supsetneq \rho_k(L_1) \supsetneq \dots \supsetneq \rho_k(L_n) = 0.$$

But this chain has length $n+1$, contradicting the fact that $\sup\{s(i) \mid i \geq 0\} = n$.

So, for any $k \geq 0$, $I_{N+k} = \rho_k(I_N)$, and $(I_i)_{i \geq 0}$ is stable.

To prove the converse of (i), let $(I_i)_{i \geq 0}$ be a bounded α -sequence of closed left ideals and suppose $\sup\{s(i) \mid i \geq 0\} = n$. If possible, let

$$I = J_0 \supsetneq J_1 \supsetneq \dots \supsetneq J_{n+1} = 0 \quad \text{be a chain}$$

of left ideals of $A(R, \alpha)$, of length $n+1$.

Let $(J_{j1})_{i \geq 0}$ denote the α -sequence $r(J_j)$ for $j = 0, \dots, n+1$.

For all $j = 0, \dots, n+1$, $(J_{ji})_{i \geq 0}$ is a bounded α -sequence, since $(I_i)_{i \geq 0}$ is bounded.

By (ii) then, the α -sequences $(I_i)_{i \geq 0}$ and $(J_{ji})_{i \geq 0}$ are all stable, and there exists $N \geq 0$ such that, for all $k \geq 0$, $\rho_k(J_{jN}) = J_{j, N+k}$ for all $j = 0, \dots, n+1$. - (1)

Suppose there exists $0 \leq j \leq n$ such that $J_{jN} = J_{j+1, N}$.

Then for all $0 \leq k \leq N$, $\alpha^{-k}(J_{jN}) = \alpha^{-k}(J_{j+1, N})$ and for all $k \geq 0$, $\rho_k(J_{jN}) = \rho_k(J_{j+1, N})$. By (1), this means that $J_{ji} = J_{j+1, i}$ for all $i \geq 0$, i.e. $J_j = J_{j+1}$, which is impossible.

Thus, for all $j = 0, \dots, n$, $J_{jN} \neq J_{j+1, N}$, so I_N contains the chain of closed left ideals

$$I_N = J_{0N} \supsetneq J_{1N} \supsetneq J_{2N} \supsetneq \dots \supsetneq J_{n+1, N} = 0.$$

This contradicts the fact that $\sup\{s(i) \mid i \geq 0\} = n$.

Therefore, I has finite length as a left $A(R, \alpha)$ -module.

Remark:

Note that the above result did not require R to be left Jordan.

4.5 Example:

Let S be a subfield of a field K and let $R = \prod_{i \in \mathbb{Z}} R_i$ where

$R_i = S$ for $i < 0$, and $R_i = K$ for $i \geq 0$.

Define $\alpha: R \rightarrow R$ by $(\alpha(r))_i = r_{i-1}$ where $r \in R$ and r_i denotes the i^{th} co-ordinate of r , for all $i \in \mathbb{Z}$.

Let I be an ideal of R and identify R_i with the ideal $(\dots, 0, R_i, 0, 0, \dots)$ where R_i appears in the i^{th} place.

If I intersects an infinite number of the R_i , say $(R_{i,n})_{n \geq 0}$, then I contains the infinite chain $(\sum_{j=1}^n R_{i,j})_{n \geq 0}$ of closed ideals.

On the other hand, if I only intersects a finite number of the R_i , say $(R_{i,n})_{n=1}^k$, then $I = \sum_{n=1}^k R_{i,n}$ and the longest possible chain of closed ideals contained in I has length k .

Note that I intersects k ($< \infty$) of the R_i iff $\alpha^{-1}(I)$ intersects k of the R_i , so an α -sequence $(I_i)_{i \geq 0}$ is bounded iff I_0 intersects only a finite number of the R_i .

Let B denote the collection of bounded α -sequences of closed left ideals of R , and let M denote the collection of maximal closed left ideals of R .

4.6 Theorem:

Let R be a left and right Jordan ring. Then $A(R, \alpha)$ is a full quotient ring iff

$$(I_i)_{i \geq 0} \in B \implies \sum_{j=0}^n \alpha^{-j}(l(I_j) \cap r(I_j)) \subseteq \sum_{M \in M} M.$$

Proof:

First note that since R is left Jordan, a left ideal I of $A(R, \alpha)$ is Artinian iff it has finite length as an $A(R, \alpha)$ -module. Thus, by lemma 4.4, I is Artinian iff $\Gamma(I)$ is a bounded α -sequence.

Assume that the condition holds, and for $i \geq 0$ denote by B_i the set

$$(I_k)_{k \geq 0} \in \mathcal{B} \quad \bigcap_{j \geq 0} \alpha^{-j}(\ell(\alpha^i(I_j)) \cap r(\alpha^i(I_j))) .$$

Let $r \in B_i$ and let I be an Artinian left ideal of $A(R, \alpha)$.

Then $r \in \bigcap_{j \geq 0} \alpha^{-j}(\ell(\alpha^i(I_j)) \cap r(\alpha^i(I_j)))$ where $(I_j)_{j \geq 0} = \Gamma(I)$.

Let $x^{-k} s x^k \in I$ for some $k \geq 0$. Then

$$\alpha^k(r) \in \ell(\alpha^i(I_k)) \cap r(\alpha^i(I_k))$$

so that $(x^{-i} r x^i)(x^{-k} s x^k) = x^{-(i+k)} \alpha^k(r) \alpha^i(s) x^{i+k}$

$$= 0 \quad \text{since } s \in I_k .$$

Similarly, $(x^{-k} s x^k)(x^{-i} r x^i) = 0$, and therefore $B_i \subseteq (\ell(I) \cap r(I))_i$ for any Artinian left ideal I of $A(R, \alpha)$, therefore $B_i \subseteq (\ell(A) \cap r(A))_i$ where A denotes the Artinian radical of $A(R, \alpha)$.

On the other hand, if $r \in (\ell(A) \cap r(A))_i$ then, clearly $r \in (\ell(I) \cap r(I))_i$ for any Artinian left ideal I of $A(R, \alpha)$.

Consequently, if $x^{-k}sx^k \in I$ then $(x^{-k}sx^k)(x^{-i}rx^i) = (x^{-i}rx^i)(x^{-k}sx^k) = 0$,

and therefore $\alpha^i(s)\alpha^k(r) = \alpha^k(r)\alpha^i(s) = 0$. But this means that

$\alpha^k(r) \in \ell(\alpha^i(I_k)) \cap r(\alpha^i(I_k))$ for all $k \geq 0$, and therefore $r \in B_i$.

Thus $B_i = (\ell(A) \cap r(A))_i$ for each $i \geq 0$.

Consider the ideal $\alpha(A)$ of $A(R, \alpha)$. Suppose it contains an infinite descending chain $(J_k)_{k \geq 0}$ of left ideals. Then, applying α^{-1} gives a strictly descending chain $(\alpha^{-1}(J_k))_{k \geq 0}$ of left ideals contained in A , contradicting the fact that A is an Artinian left ideal. Thus $\alpha(A) \subseteq A$, and similarly it can be shown that $\alpha^{-1}(A) \subseteq A$. Therefore, A is an α -stable ideal of $A(R, \alpha)$.

Now let $x^{-i}rx^i \in \ell(A) \cap r(A)$. Then

$$(x^{-i}rx^i)A = A(x^{-i}rx^i) = 0, \text{ whence}$$

$$\alpha(x^{-i}rx^i)\alpha(A) = \alpha(A)\alpha(x^{-i}rx^i) = 0. \text{ But } \alpha(A) = A,$$

so $\ell(A) \cap r(A)$ is α -invariant. A similar argument shows that $\ell(A) \cap r(A)$ is α^{-1} -invariant; hence it is α -stable. By proposition 2.4, $B_i = B_j$ for all $i, j \geq 0$, so $B_i = B_0$, i.e.

$$B_i = \bigcap_{(I_k)_{k \geq 0} \in B} \bigcap_{j \geq 0} \alpha^{-j}(\ell(I_j) \cap r(I_j)).$$

Now, by theorem 3.23, if J denotes the Jacobson radical of $A(R, \alpha)$

then $J_i = \bigcap_{M \in \mathcal{M}} M$ for all $i \geq 0$, so that $B_i \subseteq J_i$ for all $i \geq 0$.

Applying Δ to both sides then gives $\lambda(A) \cap r(A) \subseteq J$.

By Stafford's theorem ([3], theorem 13.10), $A(R, \alpha)$ is a full quotient ring.

Conversely, assume that $A(R, \alpha)$ is a full quotient ring. Then, by Stafford's Theorem, $\lambda(A) \cap r(A) \subseteq J$. It was shown above that $B_0 = (\lambda(A) \cap r(A))_i$ for all $i \geq 0$, so applying τ to both sides, along with theorem 3.23, gives $B_0 \subseteq \bigcap_{M \in \mathcal{M}} M$.

4.7 Example:

Let $S = K[x_i]_{i \in \mathbb{Z}}$, where K is a field, α the K -endomorphism such that $\alpha(x_i) = x_{i+1}$, and let R be the localization of S at the set $K[x_i]_{i \geq 1} \setminus \{0\}$.

Extend α to R by defining, for $s \in S$ and $c \in K[x_i]_{i \geq 1}$, $\alpha(sc^{-1}) = \alpha(s)\alpha(c)^{-1}$.

Now consider the full 2×2 matrix ring $M_2(R)$,

and define $\alpha_2: M_2(R) \rightarrow M_2(R)$ by $\alpha_2 \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix} = \begin{bmatrix} \alpha(r_1) & \alpha(r_2) \\ \alpha(r_3) & \alpha(r_4) \end{bmatrix}$.

By proposition 1.27, $A(R, \alpha)$ is a field, so (R, α) is left and right Jordan, and by theorem 3.2, $(M_2(R), \alpha_2)$ is left Jordan. A "right-handed" version of theorem 3.2 shows that $(M_2(R), \alpha_2)$ is also right Jordan.

Denote by I the left ideal $\begin{pmatrix} R & 0 \\ R & 0 \end{pmatrix}$ of $M_2(R)$. Since I is α_2 -stable, it is closed by proposition 2.5. Now assume that I strictly contains a non-zero closed left ideal I_1 , and let $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \in I_1$, with $a \neq 0$. Since I_1 is a left ideal, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \in I_1$, so b may also be assumed to be non-zero.

If $n \in \mathbb{N}$ is such that both $\alpha^n(a)$ and $\alpha^n(b)$ are units of R , then $\alpha_2^n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} \alpha^n(a)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \alpha_2^n \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$ and since I_1 is closed, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in I_1$.

Similarly, $\alpha_2^n \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \alpha^n(b)^{-1} \end{bmatrix} \alpha_2^n \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$, so the fact that I_1 is closed gives $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in I_1$.

Thus, $\begin{pmatrix} R & 0 \\ R & 0 \end{pmatrix} \subseteq I_1$, which contradicts the assumption that I_1 is strictly contained in I .

This means that I cannot strictly contain a non-zero closed left ideal, so that the α_2 -sequence $(I_i)_{i \geq 0}$ defined by $I_i = I$ for all $i \geq 0$, is bounded.

Therefore,

$$\begin{aligned} (I_i)_{i \geq 0}^{\alpha_2} & \subseteq \bigcap_{j \geq 0} \alpha^{-j}(\ell(I_j) \cap r(I_j)) \subseteq \bigcap_{j \geq 0} \alpha^{-j}(\ell(I) \cap r(I)) \\ & \subseteq \ell(I) \cap r(I). \end{aligned}$$

But $\ell(I) = 0$, so theorem 4.6 shows that $A(R, \alpha)$ is a full quotient ring.

Another way of seeing this, without referring to theorem 4.6, is to note that $A(R, \alpha)$ is a field, and by the proof of theorem 3.2, $A(M_2(R), \alpha_2) \cong M_2(A(R, \alpha))$, which is Artinian and therefore a quotient ring.

§2. Artinian Quotient Rings.

In view of Small's theorem ([3], theorem 2.3(c)), which states that a left Noetherian ring S has a left Artinian left quotient ring iff $C_S(0) = C_S(N(S))$, one way to approach the problem of when $A(R, \alpha)$ has a left Artinian left quotient ring would be to assume that R is left Jordan, and then to work with the regularity condition $C_{A(R, \alpha)}(0) = C_{A(R, \alpha)}(N)$.

However, there are other versions of Small's theorem which do not require S to be left Noetherian, and the object here is to use one of these variations (theorem 1.8) to avoid the necessity for R to be left Jordan.

Specifically, R will be assumed to be left Noetherian, which by corollary 2.31 immediately means that $A(R, \alpha)$ satisfies condition (ii) of theorem 1.8, namely that $N(A(R, \alpha))$ (which will just be denoted by N) is nilpotent. In addition, R will be assumed to have an α -invariant nilpotent radical - this assumption is not very restrictive,

as there are no known examples of a left Noetherian ring whose nilpotent radical is not α -invariant, and this remains an open problem.

The main consequences of the assumption that $N(R)$ is α -invariant are that $\frac{A(R,\alpha)}{N}$ can then be shown to be a semiprime left Goldie ring, and that $A(R,\alpha)$ can be shown to have finite reduced rank, as a left $A(R,\alpha)$ -module. Thus, conditions (i), (ii) and (iii) of theorem 1.8 are satisfied automatically.

Finally, again using the α -invariance of $N(R)$, the regularity condition can be translated to R , to obtain the final result: if R is left Noetherian with α -invariant nilpotent radical, then $A(R,\alpha)$ has a left Artinian left quotient ring iff $C_R(N(R)) = C_\alpha(0)$. Here, $C_\alpha(0)$ is the set $\{r \in R \mid \alpha^j(r) \in C_R(0) \text{ for all } j \geq 0\}$.

The main result is then applied to an example which is not left Jordan.

4.8 Proposition:

Let R be a left Noetherian ring with nilpotent radical $N(R)$, such that $\alpha(N(R)) \subseteq N(R)$.

Then $G(N(R)) = N$ where N denotes the nilpotent radical of $A(R,\alpha)$.

Proof:

By theorem 2.2, $G(N(R))$ is an ideal of $A(R,\alpha)$, and by Levitzki's theorem (corollary 1.8 of [3]), $N(R)$ is nilpotent. To

show that $G(N(R))$ is nilpotent, let $k \geq 0$ be such that $N(R)^k = 0$.

Let $x^{-i_j} a_j x^{i_j} \in G(N(R))$ where $a_j \in N(R)$ and $i_j \geq 0$, for each $j = 1, \dots, k$.

$$\begin{aligned} \text{Then } \prod_{j=1}^k x^{-i_j} a_j x^{i_j} &= \prod_{j=1}^k x^{-i} \alpha^{i-j} (a_j) x^i \text{ where } i = \max\{i_1, i_2, \dots, i_k\}, \\ &= x^{-i} \prod_{j=1}^k \alpha^{i-j} (a_j) x^i \\ &= 0 \text{ since } a_j \in N(R), \end{aligned}$$

$N(R)$ is α -invariant, and $N(R)^k = 0$.

Thus, $G(N(R))^k = 0$, and $G(N(R)) \subseteq N$.

Now, $N \cap R$ is a nil ideal of R , so by Levitzki's theorem, it is nilpotent, and therefore $N \cap R \subseteq N(R)$.

By the proof of theorem 2.28, N is an α -stable ideal of $A(R, \alpha)$, so by theorem 2.2, $N \cap R$ is an α -invariant ideal of R .

Applying G therefore gives $G(N \cap R) \subseteq G(N(R))$, but by theorem 2.2 this means $N \subseteq G(N(R))$.

Remark:

Any commutative ring R has an α -invariant nilpotent radical, since in that case $N(R)$ consists of all the nilpotent elements of R . Also, the result of Jategaonkar ([13], lemma 1.1) shows that left Artinian rings have α -invariant nilpotent radicals.

4.9 Lemma:

Let R be a left Noetherian ring such that $N(R)$ is α -invariant. Then $A(R, \alpha)/N$ is a semiprime left Goldie ring.

Proof:

Let k be such that $N(R)^k = 0$, and let $r \in \alpha^{-1}(N(R))$. Then $\alpha(r^k) = \alpha(r)^k \in N(R)^k = 0$, so $r^k = 0$ and $\alpha^{-1}(N(R))$ is a nilpotent ideal of R . Thus, $\alpha^{-1}(N(R)) \subseteq N(R)$, and $N(R)$ is α -stable.

Since $N(R)$ is α -stable, it is possible to define a ring monomorphism $\bar{\alpha} : R/N(R) \rightarrow R/N(R)$ by

$$\bar{\alpha}(r + N(R)) = \alpha(r) + N(R).$$

Now attempt to define a map $\psi : A(R/N(R), \bar{\alpha}) \rightarrow A(R, \alpha)/N$ by

$$\psi(x^{-i}(r+N(R))x^i) = x^{-i}rx^i + N.$$

To show that ψ is well-defined, let $r, s \in R$ be such that $x^{-j}(r+N(R))x^j = x^{-j}(s+N(R))x^j$.

$$\text{Then } x^{-(i+j)}\bar{\alpha}^{-i}(r+N(R))x^{i+j} = x^{-(i+j)}\bar{\alpha}^{-j}(s+N(R))x^{i+j},$$

$$\text{i.e. } \bar{\alpha}^i(r+N(R)) = \bar{\alpha}^j(s+N(R))$$

$$\text{or } \alpha^i(r) + N(R) = \alpha^j(s) + N(R).$$

Therefore, $\alpha^i(r) - \alpha^j(s) \in N(R)$ and

$$x^{-(i+j)}(\alpha^i(r) - \alpha^j(s))x^{i+j} \in G(N(R)).$$

But by proposition 4.8, $G(N(R)) = N$, so that $x^{-j}rx^j - x^{-i}sx^i \in N$, and ψ is well-defined.

To show that ψ is a ring homomorphism, let $r, s \in R$ and $i, j \geq 0$. Then

$$\begin{aligned} & \psi(x^{-i}(r+N(R))x^i + x^{-j}(s+N(R))x^j) \\ &= \psi(x^{-(i+j)}[\alpha^{-j}(r+N(R)) + \alpha^{-i}(s+N(R))]x^{i+j}) \\ &= \psi(x^{-(i+j)}(\alpha^j(r) + \alpha^i(s) + N(R))x^{i+j}) \\ &= x^{-(i+j)}(\alpha^j(r) + \alpha^i(s))x^{i+j} + N \\ &= (x^{-i}rx^i + N) + (x^{-j}sx^j + N), \text{ so } \psi \text{ is} \end{aligned}$$

additive. Also

$$\begin{aligned} & \psi(x^{-i}(r+N(R))x^i \cdot x^{-j}(s+N(R))x^j) \\ &= \psi(x^{-(i+j)}\alpha^{-j}(r+N(R))\alpha^{-i}(s+N(R))x^{i+j}) \\ &= \psi(x^{-(i+j)}(\alpha^j(r)\alpha^i(s) + N(R))x^{i+j}) \\ &= x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{i+j} + N \\ &= (x^{-i}rx^i + N)(x^{-j}sx^j + N), \text{ so } \psi \text{ is} \end{aligned}$$

a ring homomorphism.

To show that ψ is injective, assume $x^{-i}rx^i \in N$. By proposition 4.8, $x^{-i}rx^i \in G(N(R))$, and by theorem 2.2, $r \in G(N(R)) \cap R = \bigcup_{n \geq 0} \alpha^{-n}(N(R))$.

Since $N(R)$ is α -stable, this means $r \in N(R)$, so ψ is injective.

It is clear that ψ is surjective, hence ψ is an isomorphism.

Now, since R is left Noetherian, $R/N(R)$ is semiprime left Noetherian, and by theorem 1.42, $A(R/N(R), \bar{\alpha})$ is semiprime left Goldie. But since ψ is an isomorphism, $A(R, \alpha)/N$ is semiprime left Goldie.

4.10 Lemma:

Let R be a left Noetherian ring, and let J be an α -stable left ideal of $A(R, \alpha)$. Then $A(R, \alpha)/J$ has finite Goldie dimension.

Proof:

Assume that $A(R, \alpha)/J$ does not have finite Goldie dimension. Then there exists a sequence $(K_i)_{i \geq 0}$ of left ideals of $A(R, \alpha)$ such that $J \subsetneq K_i$ and the sum $\sum_{i=0}^{\infty} \frac{K_i}{J}$ is direct.

If $(K_{ij})_{j \geq 0}$ denotes the α -sequence $\Gamma(K_i)$ and $(J_j)_{j \geq 0}$ denotes the α -sequence $\Gamma(J)$, then by theorem 1.33, $J_j \subseteq K_{ij}$ for all $i, j \geq 0$. It is now claimed that for each $j \geq 0$, the sum $\sum_{i=0}^{\infty} \frac{K_{ij}}{J_j}$ is direct.

Indeed, for $j \geq 0$ let $r_i \in K_{ij}$ (for $i = 0, \dots, p$) be such that $\sum_{i=0}^p r_i + J_j = 0$. Then $\sum_{i=0}^p r_i \in J_j$, so that $x^{-j} \sum_{i=0}^p r_i x^j \in J$,

or $\sum_{i=0}^p x^{-j} r_i x^j \in J$.

But since $r_i \in K_{ij}$, $x^{-j} r_i x^j \in K_i$, and directness of the sum $\sum_{i=0}^{\infty} \frac{K_i}{J}$ means that $x^{-j} r_i x^j \in J$, i.e. $r_i \in J_j$ for each $i = 1, \dots, p$, so the sum $\sum_{i=0}^{\infty} \frac{K_{ij}}{J_j}$ is direct.

Now, since $J \not\subseteq K_i$, for each $i \geq 0$ there exists $z \geq 0$ with $J_z \not\subseteq K_{i2}$.

Furthermore, if $r \in K_{i2} - J_z$ then $x^{-z} r x^z \in K_i - J$, i.e. $x^{-(z+k)} a^k(r) x^{z+k} \in K_i - J$. Consequently, $a^k(r) \in K_{i, z+k} - J_{z+k}$ for all $k \geq 0$; so for each $i \geq 0$ there exists $z_0 \geq 0$ such that for all $z \geq z_0$, $\frac{K_{i2}}{J_z} \neq 0$.

Therefore, there exists $j_0 \geq 0$ such that $\frac{K_0 \cdot j_0}{J_{j_0}} \neq 0$.

By the above argument, there exists $j_1 \geq j_0$ with $\frac{K_1 \cdot j_1}{J_{j_1}} \neq 0$

and $\frac{K_0 \cdot j_1}{J_{j_1}} \neq 0$. It was shown above that the sum $\frac{K_0 \cdot j_1}{J_{j_1}} + \frac{K_1 \cdot j_1}{J_{j_1}}$ is direct.

This procedure can be repeated indefinitely to yield, for any $n \geq 0$, a direct sum

$$\frac{K_0 \cdot j_n}{J_{j_n}} \oplus \frac{K_1 \cdot j_n}{J_{j_n}} \oplus \dots \oplus \frac{K_n \cdot j_n}{J_{j_n}} \quad \text{of non-zero}$$

submodules of R/J_n . (1).

But J is an α -stable left ideal of $A(R, \alpha)$, so by proposition 2.4, $J_j = J_i$ for all $i, j \geq 0$, and in particular, $J_j = J_0 = J \cap R$ for all $j \geq 0$.

Therefore, from (1), given any $n \geq 0$ there exists a direct sum

$$\frac{K_0 J_n}{J \cap R} \oplus \frac{K_1 J_n}{J \cap R} \oplus \dots \oplus \frac{K_n J_n}{J \cap R} \quad \text{of non-zero}$$

submodules of $R/J \cap R$.

But this contradicts the fact that, since R is left Noetherian, $R/J \cap R$ is Noetherian and therefore has finite Goldie dimension.

4.11 Lemma:

Let R be a left Noetherian ring with $N(R)$ α -invariant. Then $\rho(A(R, \alpha)) < \infty$, where ρ denotes the reduced rank of a left $A(R, \alpha)$ -module.

Proof:

First, note that by corollary 2.31, N (the nilpotent radical of $A(R, \alpha)$) is nilpotent, of index k say. By lemma 4.9, $\frac{A(R, \alpha)}{N}$ is a semiprime left Goldie ring.

Then, the reduced rank of the left $A(R, \alpha)$ -module $A(R, \alpha)$ is given by

$$\rho(A(R, \alpha)) = \sum_{i=0}^{k-1} \rho \left(\frac{A(R, \alpha)}{N} \right) (N^i/N^{i+1})$$

where the reduced ranks on the right are calculated over the semiprime left Goldie ring $\frac{A(R, \alpha)}{N}$. (See definition 1.5.)

It is therefore sufficient to show that $\rho \frac{A(R, \alpha)}{N} (N^i/N^{i+1})$ is finite, for each $i = 0, \dots, k-1$.

Consider the set $C_{A(R, \alpha)}(N)$ (which will be denoted by $C(N)$ from now on) and let $c \in C(N)$, $a \in A(R, \alpha)$ with $\alpha(c)a \in N$. Then $c\alpha^{-1}(a) \in \alpha^{-1}(N) = N$ since N is α -stable, by the proof of theorem 2.28.

Thus, $\alpha^{-1}(a) \in N$, or $a \in \alpha(N) = N$, so that $\alpha(c) \in C'(N)$. A similar argument shows that $\alpha(c) \in {}^1C(N)$, therefore $\alpha(C(N)) \subseteq C(N)$. If the argument is repeated using α^{-1} instead of α , it yields that $\alpha^{-1}(C(N)) \subseteq C(N)$.

Now consider the singular submodule $Z(N^i/N^{i+1})$ of the $A(R, \alpha)/N$ -module N^i/N^{i+1} . Then, by definition,

$$\begin{aligned} Z(N^i/N^{i+1}) &= \{r + N^{i+1} \mid cr \in N^{i+1} \text{ for some } c \in C(N)\} \\ &= A_i/N^{i+1} \text{ where } A_i = \{r \in N^i \mid cr \in N^{i+1}, \text{ for} \end{aligned}$$

some $c \in C(N)\}$.

Let $r \in A_i$. Then $cr \in N^{i+1}$ for some $c \in C(N)$, so that $\alpha(c)\alpha(r) \in \alpha(N^{i+1}) = N^{i+1}$ and $\alpha^{-1}(c)\alpha^{-1}(r) \in \alpha^{-1}(N^{i+1}) = N^{i+1}$. But $\alpha(C(N)) = C(N)$, so $\alpha(c)$ and $\alpha^{-1}(c)$ are both elements of $C(N)$, consequently $\alpha(r), \alpha^{-1}(r) \in A_i$, and A_i is an α -stable left ideal of $A(R, \alpha)$.

By lemma 4.10, N^i/A_i has finite Goldie dimension, for each $i = 0, \dots, k-1$.

$$\begin{aligned} \text{Thus, } \frac{\dim_{A(R, \alpha)}(N^i/N^{i+1})}{N} &= \dim \frac{N^i/N^{i+1}}{Z(N^i/N^{i+1})} \\ &= \dim \frac{N^i/N^{i+1}}{A_i/N^{i+1}} \\ &= \dim N^i/A_i < \infty \text{ for each } i = 0, \dots, k-1. \end{aligned}$$

Notation:

Denote by $'C_\alpha(0)$ the set $\{r \in R \mid \alpha^n(r) \in 'C_R(0) \text{ for each } n \geq 0\}$, by $C'_\alpha(0)$ the set $\{r \in R \mid \alpha^n(r) \in C'_R(0) \text{ for each } n \geq 0\}$, and by $C_\alpha(0)$ the set $C'_\alpha(0) \cap 'C_\alpha(0)$.

4.12 Theorem:

Let R be a left Noetherian ring such that $N(R)$ is α -invariant. Then $A(R, \alpha)$ has a left Artinian left quotient ring iff $C_\alpha(0) = C_R(N(R))$.

Proof:

First note that, by corollary 2.31, N is nilpotent; by lemma 4.9 $\frac{A(R, \alpha)}{N}$ is a semiprime left Goldie ring, and by lemma 4.11, $A(R, \alpha)$ has finite reduced rank as a left $A(R, \alpha)$ -module.

Denoting the sets $C_{A(R, \alpha)}(0)$ and $C_{A(R, \alpha)}(N)$ by $C(0)$ and $C(N)$ respectively, it is claimed that $C(0) = C(N)$ iff $C_\alpha(0) = C_R(N(R))$.

Indeed, as in the proof of lemma 4.9, the fact that $N(R)$ is α -invariant means it is α -stable, so it is possible to define a monomorphism $\bar{\alpha}: R/N(R) \rightarrow R/N(R)$ by $\bar{\alpha}(r + N(R)) = \alpha(r) + N(R)$. But $R/N(R)$ is a semiprime left Noetherian ring, so by Goldie's theorem, it has a semisimple Artinian left quotient ring. By proposition 2.4 of [13] $\bar{\alpha}(C_{R/N(R)}(0)) \subseteq C_{R/N(R)}(0)$, so that $\alpha(C_R(N(R))) \subseteq C_R(N(R))$.

Now let $x^{-i}rx^i \in C(N)$ and let $s \in R$ be such that $rs \in N(R)$. Then $x^{-i}rsx^i \in G(N(R)) = N$ by proposition 4.8, i.e. $(x^{-i}rx^i)(x^{-i}sx^i) \in N$. Therefore $x^{-i}sx^i \in N$, and $s \in G(N(R)) \cap R = N(R)$, by theorem 2.2 and since $N(R)$ is α -stable. Hence $r \in C_R^i(N(R))$.

Similarly, it can be shown that $r \in {}^iC_R(N(R))$, so that $C(N) \subseteq \bigcup_{i \geq 0} x^{-i}C_R(N(R))x^i$.

Conversely, let $r \in C_R(N(R))$, let $i \geq 0$ and let $s \in R$, $j \geq 0$ be such that $(x^{-i}rx^i)(x^{-j}sx^j) \in N$. Then

$x^{-(i+j)} \alpha^j(r) \alpha^i(s) x^{i+j} \in N = G(N(R))$, so by theorem 2.2,

$\alpha^j(r) \alpha^i(s) \in G(N(R)) \cap R = N(R)$. Since $\alpha(C_R(N(R))) \subseteq C_R(N(R))$,

$\alpha^j(r) \in C_R(N(R))$, and therefore $\alpha^i(s) \in N(R)$. Consequently

$x^{-(i+j)} \alpha^i(s) x^{i+j} = x^{-j} s x^j \in G(N(R)) = N$, and $x^{-i} r x^i \in C'(N)$.

A similar argument shows that $x^{-i} r x^i \in C(N)$, so $x^{-i} r x^i \in C(N)$;

i.e. $\bigcup_{i \geq 0} x^{-i} C_R(N(R)) x^i = C(N)$.

By proposition 1.25, $\bigcup_{i \geq 0} x^{-i} C_\alpha(0) x^i = C(0)$, and it is now clear

that $C(0) = C(N)$ iff $C_\alpha(0) = C_R(N(R))$.

By theorem 1.8, $A(R, \alpha)$ has a left Artinian left quotient ring
iff $C(0) = C(N)$, i.e. iff $C_\alpha(0) = C_R(N(R))$.

Theorem 4.12 leads to the following corollary, which is a partial
converse to corollary 7.3 of [16].

4.13 Corollary:

Let R be a left Noetherian ring with α -invariant nilpotent
radical $N(R)$, satisfying either of the following conditions:

(i) $\alpha(C_R(0)) \subseteq C_R(0)$;

(ii) R has a Noetherian quotient ring.

Then $A(R, \alpha)$ has a left Artinian left quotient ring iff R has a left Artinian left quotient ring.

Proof:

If R satisfies condition (i) then $C_\alpha(0) = C_R(0)$, so the proof of theorem 4.12 shows that $C_R(0) = C_R(N(R))$ iff $C(0) = C(N)$. By Small's theorem (theorem 2.3 (c) of [3]), R is a left order in a left Artinian ring iff $C_R(0) = C_R(N(R))$, and by the proof of theorem 4.12, $A(R, \alpha)$ is a left order in a left Artinian ring iff $C(0) = C(N)$.

If R satisfies condition (ii), then by theorem 7.2(ii) of [16], $\alpha(C_R(0)) \subseteq C_R(0)$, i.e. R satisfies condition (i).

4.14 Example:

Consider the K -algebra endomorphism $\sigma: K[y] \rightarrow K[y]$ such that $\sigma(y) = y^2$ and let $R = M_2(K[y])$, the 2×2 upper triangular matrix ring over $K[y]$, K being a field.

$$\text{Define } \alpha: R \rightarrow R \text{ by } \alpha \begin{bmatrix} f_1 & f_2 \\ 0 & f_3 \end{bmatrix} = \begin{bmatrix} \sigma(f_1) & \sigma(f_2) \\ 0 & \sigma(f_3) \end{bmatrix}.$$

Then R is left Noetherian, and $N(R) = \begin{bmatrix} 0 & K[y] \\ 0 & 0 \end{bmatrix}$ is clearly α -invariant.

To find $C_R(0)$, let $r = \begin{bmatrix} f_1 & f_2 \\ 0 & f_3 \end{bmatrix} \in R$. If $f_1 = 0$ then r

is annihilated on the right by the element $\begin{bmatrix} g_1 & 0 \\ 0 & 0 \end{bmatrix}$ for any

$$g_1 \in K[y], \text{ hence } C_R'(0) \subseteq \left\{ \begin{bmatrix} f_1 & f_2 \\ 0 & f_3 \end{bmatrix} \in R \mid f_1 \neq 0 \right\}.$$

Similarly, if $f_3 = 0$ then r is annihilated on the left

$$\text{by } \begin{bmatrix} 0 & 0 \\ 0 & g_3 \end{bmatrix} \text{ for any } g_3 \in K[y], \text{ so that } {}^lC_R(0) \subseteq \left\{ \begin{bmatrix} f_1 & f_2 \\ 0 & f_3 \end{bmatrix} \in R \mid f_3 \neq 0 \right\}.$$

$$\text{Thus, } C_R(0) = {}^lC_R(0) \cap C_R'(0) \subseteq \left\{ \begin{bmatrix} f_1 & f_2 \\ 0 & f_3 \end{bmatrix} \in R \mid f_1 \neq 0 \neq f_3 \right\}. \quad (1)$$

Now, let $\begin{bmatrix} f_1 & f_2 \\ 0 & f_3 \end{bmatrix}$ be such that $f_1 \neq 0 \neq f_3$ and let

$$g_1, g_2, g_3 \in K[y].$$

$$\begin{aligned} \text{Then } \begin{bmatrix} f_1 & f_2 \\ 0 & f_3 \end{bmatrix} \begin{bmatrix} g_1 & g_2 \\ 0 & g_3 \end{bmatrix} &= \begin{bmatrix} f_1 g_1 & f_1 g_2 + f_2 g_3 \\ 0 & f_3 g_3 \end{bmatrix} \\ &= 0 \quad \text{iff } g_1 = g_3 = g_2 = 0. \end{aligned}$$

Similarly,

$$\begin{aligned} \begin{bmatrix} g_1 & g_2 \\ 0 & g_3 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \\ 0 & f_3 \end{bmatrix} &= \begin{bmatrix} g_1 f_1 & g_2 f_3 + g_1 f_2 \\ 0 & g_3 f_3 \end{bmatrix} \\ &= 0 \quad \text{iff } g_1 = g_3 = g_2 = 0. \end{aligned}$$

$$\text{Thus, } \begin{bmatrix} f_1 & f_2 \\ 0 & f_3 \end{bmatrix} \in C_R(0) \text{ and from (1)}$$

above, $C_R(0) = \left\{ \begin{bmatrix} f_1 & f_2 \\ 0 & f_3 \end{bmatrix} \in R \mid f_1 \neq 0 \neq f_3 \right\}.$

Since $\alpha(C_R(0)) \subseteq C_R(0)$, $C_\alpha(0) = C_R(0).$

To find $C_R(N(R))$, consider the factor ring $R/N(R)$, which

is isomorphic to the subring $\begin{bmatrix} K[y] & 0 \\ 0 & K[y] \end{bmatrix}$ of R via the map

$$\begin{bmatrix} f_1 & f_2 \\ 0 & f_3 \end{bmatrix} + N(R) \mapsto \begin{bmatrix} f_1 & 0 \\ 0 & f_3 \end{bmatrix}.$$

The regular elements of $\begin{bmatrix} K[y] & 0 \\ 0 & K[y] \end{bmatrix}$ are those of the form $\begin{bmatrix} f_1 & 0 \\ 0 & f_3 \end{bmatrix}$

for $f_1 \neq 0 \neq f_3$, so that the regular elements of $R/N(R)$ are those

having form $\begin{bmatrix} f_1 & f_2 \\ 0 & f_3 \end{bmatrix} + N(R)$ for $f_1 \neq 0 \neq f_3$. Hence

$$C_R(N(R)) = \left\{ \begin{bmatrix} f_1 & f_2 \\ 0 & f_3 \end{bmatrix} \in R \mid f_1 \neq 0 \neq f_3 \right\}.$$

Consequently, $C_\alpha(0) = C_R(N(R)).$

By theorem 4.12, $A(R, \alpha)$ has a left Artinian left quotient ring.

It should be noted that (R, α) is not left Jordan. Indeed, the isomorphism $\psi: M_n(A(R, \alpha)) \rightarrow A(M_n(R), \bar{\alpha})$ in the proof of theorem 3.2 restricts to an isomorphism $\bar{\psi}: \bar{M}_n(A(R, \alpha)) \rightarrow A(\bar{M}_n(R), \bar{\alpha})$ where \bar{M}_n

denotes the $n \times n$ upper triangular matrix ring.

Thus, in this example, $A(R, \alpha) \cong M_2(A(K[y], \sigma))$. By example 1.41, $A(K[y], \sigma)$ is not Noetherian, hence neither is $M_2(A(K[y], \sigma))$, so that (R, α) is not left Jordan.

CHAPTER 5.

APPLICATIONS AND EXAMPLES.

As was seen in chapter 1, it is possible to study several properties of the skew Laurent polynomial ring $R[x, x^{-1}, \alpha]$ when α is an automorphism. Specifically, Jordan [15] found conditions under which $R[x, x^{-1}, \alpha]$ is semiprimitive, primitive, and Jacobson. These results appear in chapter 1 as theorems 1.17, 1.19, 1.20 and 1.22. In [17] he found necessary and sufficient conditions for $R[x, x^{-1}, \alpha]$ to be simple (see theorem 1.24).

The aim of this chapter is to generalize these results to the case where α is assumed only to be a monomorphism. This is achieved by using the fact that, as seen in remark 1.29, $A(R, \alpha)[x, x^{-1}, \alpha] = R[x, x^{-1}, \alpha]$, and that $\alpha: A(R, \alpha) \rightarrow A(R, \alpha)$ is an automorphism.

As in [15], two separate sets of conditions are found which are sufficient for $R[x, x^{-1}, \alpha]$ to be primitive. Since they are logically independent (as shown in [15]), for the case where α is an automorphism, they are logically independent here too.

The final part of the chapter gives two examples which arise from earlier work. One is an example of a ring R and a monomorphism $\alpha: R \rightarrow R$ such that $\alpha(C_R(0)) \not\subseteq C_R(0)$, the second is an example where $A(R, \alpha)$ is a commutative domain, each factor ring of which is uniform, and yet $A(R, \alpha)$ does not have Krull dimension.

First, we extend the definitions of section 4 in chapter 1 to the case where α is only a monomorphism.

§1. Primitivity and Semiprimitivity of $R[x, x^{-1}, \alpha]$

5.1 Definition:

Let R be a ring, $\alpha: R \rightarrow R$ a monomorphism. Then, an α -stable ideal I of R is called α -prime if, for all α -stable ideals A, B of R , $AB \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$.

R is called an α -prime ring if 0 is an α -prime ideal.

5.2 Proposition:

If P is an α -prime ideal of $A(R, \alpha)$ then $P \cap R$ is an α -prime ideal of R .

Proof:

Let I and J be α -stable ideals of R . Then $G(I)G(J) \subseteq G(IJ)$.

Indeed, any element a of $G(I)G(J)$ has the form

$$a = \sum_{j=1}^n (x^{-i_j} a_j x^{i_j}) (x^{-k_j} b_j x^{k_j}) \text{ where } a_j \in I, b_j \in J, \text{ and}$$

$i_j, k_j \geq 0$ for each $j = 1, \dots, n$.

Let $k = \max\{i_j, k_j \mid j = 1, \dots, n\}$ so that

$$a = x^{-k} \left[\sum_{j=1}^n x^{k-i_j} a_j x^{k-k_j} b_j \right] x^k.$$

But I and J are α -stable, therefore $\alpha^{k-i_j}(a_j) \in I$ and $\alpha^{k-k_j}(b_j) \in J$ for each $j = 1, \dots, n$, hence $a \in G(IJ)$.

Now let P be an α -prime ideal of $A(R, \alpha)$ and let I, J be α -stable ideals of R with $IJ \subseteq P \cap R$. By remark 2.3(ii), $P \cap R$ is an α -stable ideal of R .

Then, by theorem 2.2, $G(IJ) \subseteq G(P \cap R) = P$, and by the above, $G(I)G(J) \subseteq P$. But P is α -prime, $G(I)$ and $G(J)$ are α -stable (by theorem 2.2) so either $G(I) \subseteq P$ or $G(J) \subseteq P$, i.e. either $G(I) \cap R \subseteq P \cap R$ or $G(J) \cap R \subseteq P \cap R$. Since I and J are α -stable, $G(I) \cap R = I$ and $G(J) \cap R = J$, whence $P \cap R$ is α -prime.

5.3 Theorem:

Let R be an α -prime, left Jordan ring. Then $R[x, x^{-1}, \alpha]$ is semiprimitive.

Proof:

Let I, J be two α -stable ideals of $A(R, \alpha)$ with $IJ = 0$. Then clearly $(I \cap R)(J \cap R) = 0$, and, by remark 2.3(ii), $I \cap R$ and $J \cap R$ are both α -stable ideals of R . Since R is α -prime, either $I \cap R = 0$ or $J \cap R = 0$. Applying G to both sides yields, by theorem 2.2, $I = 0$ or $J = 0$, so that $A(R, \alpha)$ is an α -prime ring.

Since R is left Jordan, $A(R, \alpha)$ is left Noetherian, and by proposition 1.17, $A(R, \alpha)[x, x^{-1}, \alpha]$ is a semiprimitive ring. Thus $R[x, x^{-1}, \alpha] = A(R, \alpha)[x, x^{-1}, \alpha]$ is semiprimitive.

5.4 Corollary:

R is α -prime iff $A(R, \alpha)$ is α -prime.

Proof:

If R is α -prime then, by the proof of theorem 5.3, $A(R, \alpha)$ is α -prime.

On the other hand, if $A(R, \alpha)$ is α -prime, then 0 is an α -prime ideal of $A(R, \alpha)$, and by proposition 5.2, 0 is an α -prime ideal of R . Hence R is α -prime.

5.5 Examples:

(i) Let K be a field, $\sigma: K \rightarrow K$ a monomorphism which is not surjective, and define $\alpha: K \oplus K \rightarrow K \oplus K$ by $\alpha(x, y) = (y, \sigma(x))$. The only proper ideals of $R = K \oplus K$ are $(0, K)$ and $(K, 0)$, neither of which are α -stable. Thus R is α -prime, but not prime, and is left and right Jordan because it is Artinian, by proposition 3.1.

By theorem 5.3 then, $R[x, x^{-1}, \alpha]$ is semiprimitive.

(ii) If R is a simple Artinian ring and $\alpha: R \rightarrow R$ is any monomorphism, then $R[x, x^{-1}, \alpha]$ is semiprimitive. Indeed, since R is prime, it is α -prime, and since R is left Artinian, it is left Jordan, by proposition 3.1. By theorem 5.3, $R[x, x^{-1}, \alpha]$ is semiprimitive.

5.6 Theorem:

Let $\alpha: R \rightarrow R$ be a monomorphism such that

(i) There exists $r \in R$ such that for each $n, k \geq 0$, $\alpha^n(r) \in C(R)$ and $\alpha^n(r) - \alpha^k(r) \in C_R(0)$ for $n \neq k$;

(ii) There exists a maximal α -sequence $(M_i)_{i \geq 0}$ of closed left ideals of R such that, for any non-zero α -stable ideal I of R , there exists $i \geq 0$ with $I \not\subseteq M_i$.

Then $R[x, x^{-1}, \alpha]$ is left primitive.

Proof:

Condition (i) implies that α is stiff on $A(R, \alpha)$.

Indeed, let $r \in R$ satisfy (i) and let $a \in R$, $j \geq 0$.

$$\begin{aligned} \text{Then } rx^{-j}ax^j &= x^{-j}\alpha^j(r)ax^j \\ &= x^{-j}\alpha^j(r)x^j \quad \text{since } \alpha^j(r) \in C(R) \\ &= x^{-j}ax^jr. \end{aligned}$$

Thus, $r \in C(A(R, \alpha))$. Also, since for any $n, k \geq 0$, ($n \neq k$), $\alpha^n(r) - \alpha^k(r) \in C_R(0)$, $\alpha^k(\alpha^m(r) - r) \in C_R(0)$ for all $m \geq 1$. By proposition 1.25, $\alpha^m(r) - r$ is a regular element of $A(R, \alpha)$ for each $m \geq 1$, so by lemma 1(i) of [15], the automorphism $\alpha: A(R, \alpha) \rightarrow A(R, \alpha)$ is stiff.

Condition (ii) implies that $A(R, \alpha)$ is α -primitive. To see this, assume that M is a maximal left ideal of $A(R, \alpha)$ which contains a non-zero α -stable ideal J of $A(R, \alpha)$, and denote the α -sequences $\Gamma(M)$ and $\Gamma(J)$ by $(M_i)_{i \geq 0}$ and $(J_i)_{i \geq 0}$ respectively.

Since J is α -stable, by proposition 2.4, $J_i = J_0$ for all $i \geq 0$, i.e. $J_i = J \cap R$ for all $i \geq 0$.

Since $J \subseteq M$, applying Γ gives $\Gamma(J) \subseteq \Gamma(M)$, i.e. $J \cap R \subseteq M_i$ for all $i \geq 0$, so that $\Gamma(M)$ does not satisfy condition (ii) of the theorem, $J \cap R$ being an α -stable ideal by remark 2.3 (ii).

Therefore, there must exist a maximal left ideal M of $A(R, \alpha)$ which does not contain a non-zero α -stable ideal - i.e., $A(R, \alpha)$ is α -primitive.

By theorem 1.19, $A(R, \alpha)[x, x^{-1}, \alpha]$ is left primitive, hence $R[x, x^{-1}, \alpha]$ is left primitive.

5.7 Corollary:

$A(R, \alpha)$ is α -primitive iff R satisfies condition (ii) of the theorem.

Proof:

If R satisfies condition (ii), then $A(R, \alpha)$ is α -primitive, by the proof of theorem 5.6.

Conversely, assume that $A(R, \alpha)$ is α -primitive, and let M be a maximal left ideal of $A(R, \alpha)$ which contains no non-zero α -stable ideal. Denote the α -sequence $\Gamma(M)$ by $(M_i)_{i \geq 0}$ and if possible, let I be a non-zero α -stable ideal of R with $I \subseteq M_i$ for each $i \geq 0$.

Since I is α -stable, by proposition 2.5, I is closed, so the sequence $(I_i)_{i \geq 0}$ where $I_i = I$ for all $i \geq 0$ is an α -sequence of

closed left ideals of R . Evidently, $\Delta(I_i)_{i \geq 0}$ is an α -stable ideal of $A(R, \alpha)$, and since $I \subseteq M_i$ for all $i \geq 0$, $\Delta(I_i)_{i \geq 0} \subseteq M$, contradicting the fact that M does not contain a non-zero α -stable ideal.

Thus, for any non-zero α -stable ideal I of R , $I \not\subseteq M_i$ for some $i \geq 0$.

5.8 Examples:

(i) Let K be a field, $K[y]$ the ring of polynomials in the indeterminate y , and let $\sigma_1: K[y] \rightarrow K[y]$ be the K -algebra endomorphism such that $\sigma_1(y) = y^2$.

Let $Q[y]$ be the quotient field of $K[y]$, i.e. the field of rational functions in y , and define $\sigma: Q[y] \rightarrow Q[y]$ by $\sigma(f/g) = \sigma_1(f)/\sigma_1(g)$. Then σ is a monomorphism on $Q[y]$.

Now let $R = Q[y] \oplus Q[y]$ and define $\alpha: R \rightarrow R$ by $\alpha(x, y) = (y, \sigma(x))$.

The only proper ideals of R are $(Q[y], 0)$ and $(0, Q[y])$, both of which are annihilator ideals and therefore, by proposition 1.34, closed. Thus, the only proper α -sequences of closed ideals are $(I_i)_{i \geq 0}$ and $(J_i)_{i \geq 0}$ where

$$I_i = \begin{cases} (Q[y], 0) & \text{for } i \text{ even} \\ (0, Q[y]) & \text{for } i \text{ odd;} \end{cases}$$

$$\text{and } J_i = \begin{cases} (0, Q[y]) & \text{for } i \text{ even} \\ (Q[y], 0) & \text{for } i \text{ odd.} \end{cases}$$

Both are maximal α -sequences and neither $(Q[y], 0)$ nor $(0, Q[y])$ contains a non-zero α -stable ideal of R .

Thus, condition (ii) of theorem 5.6 is satisfied; consequently $A(R, \alpha)$ is α -primitive.

Now consider the element (y^3, y^2) of R .

If n is odd, then $\alpha^n(y^3, y^2) = (y^{2^{(n+1/2)}}, y^{3 \cdot 2^{(n+1/2)}})$ and
if n is even, then $\alpha^n(y^3, y^2) = (y^{3 \cdot 2^{(n/2)}}, y^{2 \cdot 2^{(n/2)}})$.

Thus, $\alpha^n(y^3, y^2) - \alpha^k(y^3, y^2)$ has a non-zero entry in both the first and the second places, for $n \neq k$, and is therefore regular, for all $n, k \geq 0$ with $n \neq k$.

By theorem 5.6, the skew Laurent polynomial ring $R[x, x^{-1}, \alpha]$ is left primitive.

(ii) Note that, since R is Artinian in the above example, it is left and right Jordan. The following is an example where R is not (left) Jordan.

Let K be a field, S a subfield of K , and assume both to be of characteristic zero.

Define $R = \prod_{i \in \mathbb{Z}} R_i$ where $R_i = \begin{cases} S & (i < 0) \\ K & (i \geq 0) \end{cases}$ and define $\alpha: R \rightarrow R$ by $(\alpha(r))_i = r_{i-1}$ where r_i denotes the i^{th} co-ordinate of the element r of R , for all $i \in \mathbb{Z}$.

For each $i \geq 0$, denote by M_i the ideal of R obtained by putting $R_i = 0$ in the product. Then M_i is a maximal ideal and, for any $n \in \mathbb{N}$, $R\alpha^n(M_i) = M_{i+n}$. But $\alpha^{-n}(M_{i+n}) = M_i$, so M_i is closed for each $i \geq 0$, and $(M_i)_{i \geq 0}$ is an α -sequence of closed ideals. By the first part of the proof of lemma 3.21, (which did not require R to be left Jordan), $(M_i)_{i \geq 0}$ is a maximal α -sequence of closed ideals.

Now let I be a non-zero α -stable ideal of R , and let $0 \neq r \in I$, with j^{th} component r_j non-zero. If $j \geq 0$ then $r \notin M_j$ and if $j < 0$ then $\alpha^{-j}(r) \notin M_0$.

Thus, $I \not\subseteq M_j$ for some $j \geq 0$, and condition (ii) of theorem 5.6 is satisfied, i.e. $A(R, \alpha)$ is α -primitive.

Now let $r \in R$ be defined by $r_i = \begin{cases} i & \text{for } i < 0 \\ i+1 & \text{for } i \geq 0 \end{cases}$.

Then, $\alpha^n(r) - \alpha^k(r)$ is regular for all $n, k \geq 0$ with $n \neq k$.

Theorem 5.6 now shows that $R[x, x^{-1}, \alpha]$ is primitive.

To see that R is not Jordan, for each $j \geq 0$ let A_j be the ideal $\prod_{i \in \mathbb{Z}} I_i$, where $I_i = \begin{cases} R_i & \text{for } i \leq j \\ 0 & \text{for } i > j \end{cases}$. Then $(A_j)_{j \geq 0}$ is an infinite ascending chain of closed ideals of R .

5.9 Theorem:

Let R be an α -prime, left Jordan ring such that

- (1) There exists $r \in R$ such that, for each $n, k \geq 0$, with

$n \neq k$, $\alpha^n(r) \in C(R)$ and $\alpha^n(r) - \alpha^k(r) \in C_R(0)$.

(ii) The intersection of all the non-zero α -prime ideals of R is non-zero.

Then $R[x, x^{-1}, \alpha]$ is left primitive.

Proof:

As in the proof of theorem 5.6, the first condition implies that α is stiff on $A(R, \alpha)$.

Now let $\{P_\lambda \mid \lambda \in \Delta\}$ be the collection of all the non-zero α -prime ideals of $A(R, \alpha)$. By proposition 5.2, for each $\lambda \in \Delta$, $P_\lambda \cap R$ is an α -prime ideal of R , non-zero by theorem 2.2.

Thus, $(\bigcap_{\lambda \in \Delta} P_\lambda) \cap R = \bigcap_{\lambda \in \Delta} (P_\lambda \cap R) \neq 0$, and consequently,

$\bigcap_{\lambda \in \Delta} P_\lambda \neq 0$. Therefore, by corollary 5.4, $A(R, \alpha)$ is an αG -ring, and by theorem 1.20, $A(R, \alpha)[x, x^{-1}, \alpha]$ is left primitive, i.e. $R[x, x^{-1}, \alpha]$ is left primitive.

5.10 Corollary:

$A(R, \alpha)$ is an αG -ring iff R is an α -prime ring such that the intersection of all the non-zero α -prime ideals of R is non-zero.

Proof:

By corollary 5.4, $A(R, \alpha)$ is α -prime iff R is α -prime, and by the proof of theorem 5.9, if R is an α -prime ring with the intersection of all the non-zero α -prime ideals non-zero, then $A(R, \alpha)$ is

an α G-ring. It is therefore sufficient to show that if $A(R, \alpha)$ is an α G-ring, then $\bigcap_{\lambda \in \Delta} P_\lambda \neq 0$ where $\{P_\lambda \mid \lambda \in \Delta\}$ is the collection of all the non-zero α -prime ideals of R .

Let $P \neq 0$ be an α -prime ideal of R , and let I, J be α -stable ideals of $A(R, \alpha)$ with $IJ \subseteq G(P)$. Then, $(I \cap R)(J \cap R) \subseteq IJ \cap R \subseteq G(P) \cap R$, and by theorem 2.2, $G(P) \cap R = P$, since P is α -stable.

But by remark 2.3(ii), $I \cap R$ and $J \cap R$ are both α -stable ideals of R , so either $I \cap R \subseteq P$ or $J \cap R \subseteq P$.

Applying G then yields that either $I \subseteq G(P)$ or $J \subseteq G(P)$; thus $G(P)$ is a non-zero α -prime ideal.

Now, if $\bigcap_{\lambda \in \Delta} P_\lambda = 0$ then since $P_\lambda = G(P_\lambda) \cap R$ for each $\lambda \in \Delta$, $\bigcap_{\lambda \in \Delta} G(P_\lambda) \cap R = (\bigcap_{\lambda \in \Delta} G(P_\lambda)) \cap R = 0$. But $\bigcap_{\lambda \in \Delta} G(P_\lambda)$ is an α -stable ideal of $A(R, \alpha)$, so by theorem 2.2, $G((\bigcap_{\lambda \in \Delta} G(P_\lambda)) \cap R) = \bigcap_{\lambda \in \Delta} G(P_\lambda) = 0$. This contradicts the fact that $A(R, \alpha)$ is an α G-ring.

5.11 Theorem:

Let R be a left Jordan ring and let $r \in R$ be such that

- (i) for each $n \geq 0$, $\alpha^n(r) \in C(R)$;
- (ii) for each $n > 0$ there exists $k \geq 0$ such that $\alpha^{n+k}(r) - \alpha^k(r)$ is a unit of R .

Then $R[x, x^{-1}, \alpha]$ is a Jacobson ring.

Proof:

As in the proof of theorem 5.6, condition (i) implies that $r \in C(A(R, \alpha))$.

Condition (ii) implies that for each $n > 0$, $\alpha^k(\alpha^n(r) - r)$ is a unit of R for some $k \geq 0$, so by proposition 1.27, $\alpha^n(r) - r$ is a unit of $A(R, \alpha)$.

By lemma 1(ii) of [15], α is rigid on $A(R, \alpha)$, and by theorem 1.22, $A(R, \alpha)[x, x^{-1}, \alpha] = R[x, x^{-1}, \alpha]$ is a Jacobson ring.

5.12 Examples:

(i) Let F be a field, $\sigma: F \rightarrow F$ a ring monomorphism which is not surjective, and assume that there exists an element r of F which has an infinite orbit under σ . (For example, the element y in the field of rational functions in y , under the monomorphism σ of example 9 has an infinite orbit.)

Let $R = M_n(F)$ for $n \geq 1$ and define $\alpha: R \rightarrow R$ by $(\alpha(s))_{ij} = \alpha(s_{ij})$ where s_{ij} denotes the (i, j) -entry of the matrix s .

Since R is simple Artinian, example 5.5(ii) shows $R[x, x^{-1}, \alpha]$ is semiprimitive, but in fact $R[x, x^{-1}, \alpha]$ is left primitive.

Indeed, since R is a simple ring, it is both α -prime and has the property that the intersection of all the non-zero α -prime ideals is non-zero. R is left Jordan because it is left Artinian, by proposition 3.1.

Also, for any $n, k \geq 0$,

$$\alpha^n \begin{bmatrix} r & & 0 \\ & r & \\ 0 & & r \end{bmatrix} \in C(R)$$

and for $n \neq k$,

$$\alpha^n \begin{bmatrix} r & & 0 \\ & r & \\ 0 & & r \end{bmatrix} - \alpha^k \begin{bmatrix} r & & 0 \\ & r & \\ 0 & & r \end{bmatrix} = \begin{bmatrix} \sigma^n(r) - \sigma^k(r) & & 0 \\ & \sigma^n(r) - \sigma^k(r) & \\ 0 & & \sigma^n(r) - \sigma^k(r) \end{bmatrix} \in C_R(0) \text{ since } \sigma^n(r) \neq \sigma^k(r),$$

r having infinite order under σ .

By theorem 5.9, then, $R[x, x^{-1}, \alpha]$ is left primitive.

(ii) Let F_i be a field for each $i = 1, \dots, n$, $\sigma_i: F_i \rightarrow F_i$ a monomorphism which is not surjective, and $r_i \in F_i$ an element with an infinite orbit under σ_i . Extend each σ_i to a monomorphism α_i on $M_{k_i}(F_i)$ in the same way as in example (i) above.

Let $R = M_{k_1}(F_1) \otimes \dots \otimes M_{k_n}(F_n)$ and define $\alpha: R \rightarrow R$ by

$$\alpha(s_1, s_2, \dots, s_n) = (\alpha_1(s_1), \alpha_2(s_2), \dots, \alpha_n(s_n)).$$

Denote by r the element $\begin{bmatrix} r_1 & & 0 \\ & r_1 & \\ 0 & & r_1 \end{bmatrix}, \begin{bmatrix} r_2 & & 0 \\ & r_2 & \\ 0 & & r_2 \end{bmatrix}, \dots, \begin{bmatrix} r_n & & 0 \\ & r_n & \\ 0 & & r_n \end{bmatrix}$

Clearly $\alpha^k(r)$ is central for each $k \geq 0$, and for all $k \geq 1$, $\alpha^k(r) - r$ is a unit of R .

By theorem 5.11, $R[x, x^{-1}, \alpha]$ is a Jacobson ring.

§2. Simplicity of $R[x, x^{-1}, \alpha]$.

By theorem 1.24, if $\alpha: R \rightarrow R$ is an automorphism then $R[x, x^{-1}, \alpha]$ is simple iff R has no proper α -ideals and $\alpha: R \rightarrow R$ is not power-inner. In this section, the ring $A(R, \alpha)$ is used to extend this result to the case where $\alpha: R \rightarrow R$ is a monomorphism - i.e. α is not necessarily surjective. In fact, $R[x, x^{-1}, \alpha]$ is found to be simple iff R has no proper α -stable ideals, and α is not a power-inner automorphism of R .

By regarding $R[x, x^{-1}, \alpha]$ as a graded ring, with n^{th} homogeneous component $A(R, \alpha)x^n$ for each $n \in \mathbb{Z}$, it can then be seen how much simplicity remains when the power-inner condition is dropped.

5.13 Theorem:

Let $\alpha: R \rightarrow R$ be a monomorphism. Then $R[x, x^{-1}, \alpha]$ is a simple ring iff

- (i) R has no proper α -stable ideals and
- (ii) α is not a power-inner automorphism of R .

Proof:

If I is a proper α -stable ideal of R then, by theorem 2.2, $G(I)$ is a proper α -stable ideal of $A(R, \alpha)$. Also, if J is a proper α -stable ideal of $A(R, \alpha)$, then by remark 2.3(ii), $J \cap R$ is a proper α -stable of R .

Therefore, condition (i) is equivalent to $A(R, \alpha)$ not having any proper α -stable ideals. In view of this, and since $R[x, x^{-1}, \alpha] = A(R, \alpha)[x, x^{-1}, \alpha]$, by theorem 1.24 it will be sufficient to show that $\alpha: R \rightarrow R$ is a power-inner automorphism iff $\alpha: A(R, \alpha) \rightarrow A(R, \alpha)$ is a power-inner automorphism.

If $\alpha: R \rightarrow R$ is a power-inner automorphism then $A(R, \alpha) = R$ and the fact that $\alpha: A(R, \alpha) \rightarrow A(R, \alpha)$ is power-inner follows trivially.

Conversely, assume that $\alpha: A(R, \alpha) \rightarrow A(R, \alpha)$ is power-inner, and let $n \in \mathbb{N}$, $x^{-i}rx^i \in A(R, \alpha)$ be such that $\alpha^n(s) = (x^{-i}rx^i)s(x^{-i}rx^i)^{-1}$ for each $s \in A(R, \alpha)$.

By 1.27, it may be assumed that i is chosen so that r is a unit of R .

Now, $\alpha^n(x^{-i}rx^i) = x^{-i}rx^i$ and since (by definition) $\alpha^n(x^{-i}rx^i) = x^{-i}\alpha^n(r)x^i$, $\alpha^n(r) = r$.

Using the fact that $(x^{-i}rx^i)^{-1} = x^{-i}r^{-1}x^i$, a similar argument shows that $\alpha^n(r^{-1}) = r^{-1}$.

Let $k \geq 1$ be such that $kn \geq i$, and let $s \in R$. Then $\alpha^i(\alpha^{kn-i}(r)s\alpha^{kn-i}(r^{-1})) = \alpha^{kn}(r)\alpha^i(s)\alpha^{kn}(r^{-1}) = r\alpha^i(s)r^{-1}$, since both r and r^{-1} are fixed under α^n .

$$\begin{aligned} \text{Therefore } x^{-i}\alpha^i(\alpha^{kn-i}(r)s\alpha^{kn-i}(r^{-1}))x^i &= x^{-i}r\alpha^i(s)r^{-1}x^i \\ &= (x^{-i}rx^i)x^{-i}\alpha^i(s)x^i(x^{-i}r^{-1}x^i) \\ &= \alpha^n(s). \end{aligned}$$

Thus, for any $s \in R$, $\alpha^{kn-i}(r)s(\alpha^{kn-i}(r))^{-1} = \alpha^n(s)$, and $\alpha: R \rightarrow R$ is a power-inner automorphism.

Remark:

It was shown in [16] that $\alpha: R \rightarrow R$ is an inner automorphism iff α is inner on $A(R, \alpha)$.

5.14 Example:

Let K be a field, $K[x_i]_{i \in \mathbb{Z}}$ the polynomial ring in the indeterminates $\{x_i \mid i \in \mathbb{Z}\}$ and let R be the localization of $K[x_i]_{i \in \mathbb{Z}}$ at the set $K[x_i]_{i \geq 0} - \{0\}$.

As before, define $\alpha: R \rightarrow R$ to be the K -endomorphism of R such that $\alpha(x_i) = x_{i+1}$, for all $i \in \mathbb{Z}$.

Then R has no proper α -invariant ideals, and α is not even an automorphism. By theorem 5.13, $R[x, x^{-1}, \alpha]$ is simple.

Recall that a ring R is called a graded ring if it is a direct sum of additive subgroups R_q (where $q \in \mathbb{Z}$) such that for $q, p \in \mathbb{Z}$, $R_p R_q \subseteq R_{p+q}$.

An element is said to be homogeneous if it belongs to R_q for some $q \in \mathbb{Z}$ and homogeneous of degree q if it is non-zero and belongs to R_q . Each element f of R can therefore be written uniquely as a sum of homogeneous elements; these are called the homogeneous components of f .

Clearly, if α is an automorphism, then $R[x, x^{-1}, \alpha]$ may be regarded as a graded ring, with q^{th} homogeneous component $R_q = Rx^q$ for $q \in \mathbb{Z}$.

Now, in the case where α is only a monomorphism, $R[x, x^{-1}, \alpha] = A(R, \alpha)[x, x^{-1}, \alpha]$ so $R[x, x^{-1}, \alpha]$ may be regarded as a graded ring with q^{th} homogeneous component $A(R, \alpha)x^q$.

An ideal I of a graded ring R is said to be homogeneous if $f \in I$ implies that each homogeneous component of f is also an element of I . The following result about homogeneous ideals is standard.

5.15 Lemma:

An ideal I of a graded ring R is homogeneous iff it is generated by a collection of homogeneous elements.

Proof:

Theorem 7, p.151 of [22] (Vol. II).

5.16 Theorem:

$R[x, x^{-1}, \alpha]$ has no proper homogeneous ideals iff R has no proper α -stable ideals.

Proof:

It is sufficient to show that $R[x, x^{-1}, \alpha]$ (which will be denoted by T) has no proper homogeneous ideals iff $A(R, \alpha)$ has no proper α -stable ideals, for as shown in the proof of theorem 5.13, this is equivalent to R having no proper α -stable ideals.

Assume that T has no proper homogeneous ideals, and if possible let J be a proper α -stable ideal of $A(R, \alpha)$.

Since J consists of homogeneous elements (of degree 0), TJT is a homogeneous ideal of T , by lemma 5.15. It is clear that TJT is non-zero.

If $1 \in TJT$ then $1 = \sum_{i=1}^k t_i j_i s_i$ where $s_i, t_i \in T$ and $j_i \in J$ for each $i = 1, \dots, k$. For each $i = 1, \dots, k$, write $t_i = \sum_n t_{i,n}$ and $s_i = \sum_m s_{i,m}$ where $t_{i,n}$ and $s_{i,n}$ are the homogeneous components of t_i and s_i of degree n and m respectively.

$$\begin{aligned} \text{Then } 1 &= \sum_{i=1}^k \left(\sum_n t_{i,n} \right) j_i \left(\sum_m s_{i,m} \right) \\ &= \sum_{i=1}^k \sum_n \sum_m t_{i,n} j_i s_{i,m}. \end{aligned}$$

Each term in this sum is homogeneous of degree $m+n$, so if $p \neq 0$,

$$\sum_{i=1}^k \sum_{m+n=p} t_{i,n} j_i s_{i,m} = 0$$

$$\text{i.e. } 1 = \sum_{i=1}^k \sum_{n=-m} t_{i,n} j_i s_{i,m}$$

$$\text{so that } 1 = \sum_{i=1}^k \sum_n r_{i,n} x^n j_i q_{i,n} x^{-n} \quad (1)$$

where $r_{i,n}, q_{i,n} \in A(R, \alpha)$, $r_{i,n} x^n = t_{i,n}$ and $q_{i,n} x^{-n} = s_{i,-n}$.

But $j_i q_{i,n} \in J$ for each $i = 1, \dots, k$, and for any $n \in \mathbb{Z}$,

$$r_{i,n} x^n j_i q_{i,n} x^{-n} = r_{i,n} \alpha^n(j_i q_{i,n}) \in J \quad \text{since } J \text{ is } \alpha\text{-stable.}$$

Thus, $1 \in J$. Since J is a proper ideal of $A(R, \alpha)$, $1 \notin TJT$, and TJT is a proper homogeneous ideal of T .

Conversely, suppose that $A(R, \alpha)$ does not have a proper α -stable ideal, and if possible let J be a proper homogeneous ideal of $R[x, x^{-1}, \alpha]$.

Let $0 \neq f \in J$. Since J is homogeneous, each homogeneous component f_n of f is also an element of J . Let $m \in \mathbb{Z}$ be such that $f_m \neq 0$.

Then $x^{-m} f_m \in J \cap A(R, \alpha)$, so $J \cap A(R, \alpha)$ is a non-zero ideal of $A(R, \alpha)$, and since J is proper, $1 \notin J \cap A(R, \alpha)$.

Now, if $r \in J \cap A(R, \alpha)$, then $\alpha^{-1}(r) = x^{-1} r x \in J \cap A(R, \alpha)$ and $\alpha(r) = x r x^{-1} \in J \cap A(R, \alpha)$. Thus $J \cap A(R, \alpha)$ is a proper α -stable ideal of $A(R, \alpha)$.

§3. Examples:

The purpose of this section is to present two examples which arise in a fairly straightforward manner from previous work.

The first, a modification of example 2.25, is an example where the image of a regular element under the monomorphism α is not

necessarily regular. By [16], theorem 7.2(ii), this means that the ring in question, which is commutative, does not have a Noetherian quotient ring.

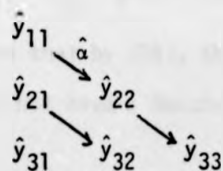
The second example uses the ideas of example 3.16 and lemma 4.10 to give a case in which $A(R, \alpha)$ is a commutative domain, every factor ring of $A(R, \alpha)$ is uniform, and yet $A(R, \alpha)$ does not have Krull dimension.

5.17 Example:

The following is an example of a ring R and a monomorphism $\alpha: R \rightarrow R$ such that $\alpha(C_R(0)) \notin C_R(0)$.

Let K be a field and let $\hat{Y} = \{\hat{y}_{ij} \mid j, i \in \mathbb{N}, j \leq i\}$ be a collection of commuting indeterminates. Let \hat{R} be the polynomial ring $K[\hat{Y}]$, and let $\hat{\alpha}: \hat{R} \rightarrow \hat{R}$ be the K -endomorphism of \hat{R} such that $\hat{\alpha}(\hat{y}_{ij}) = \hat{y}_{i+1, j+1}$, for all $i, j \in \mathbb{N}$ with $j \leq i$.

As in example 2.25, the action of $\hat{\alpha}$ can be represented in the array:



Now let I be the ideal of \hat{R} generated by the set $\{\hat{y}_{ik}\hat{y}_{il} \mid i, k, l \in \mathbb{N}, k \neq l\}$.

Any element g of I is a finite sum of polynomials of the form $f\hat{y}_{ik}\hat{y}_{il}$ ($i, k, l \in \mathbb{N}, k \neq l, f \in \hat{R}$) so $\hat{\alpha}(g)$ is a finite sum of elements of the form $\hat{\alpha}(f)\hat{y}_{i+1, k+1}\hat{y}_{i+1, l+1}$.

Thus, I is $\hat{\alpha}$ -invariant.

Now let $f \in \hat{R}$ be such that $\hat{\alpha}(f) \in I$. Since $\hat{\alpha}(\hat{R}) = K[\hat{Z}]$ where $\hat{Z} = \{\hat{y}_{ij} \mid i, j \geq 2, j \leq i\} \subseteq \hat{Y}$, $\hat{\alpha}(f) \in K[\hat{Z}]$ and $\hat{\alpha}(f)$ is a finite sum of terms of the form $g\hat{y}_{ik}\hat{y}_{il}$ where $g \in K[\hat{Z}]$, $i, k, l \geq 2$ and $k \neq l$.

Therefore, f is a finite sum of terms of the form $h\hat{y}_{i-1, k-1}\hat{y}_{i-1, l-1}$ where $\hat{\alpha}(h) = g$. Thus, $f \in I$ and I is $\hat{\alpha}$ -stable.

Now let $R = \hat{R}/I$ and define $\alpha: R \rightarrow R$ by $\alpha(f+I) = \hat{\alpha}(f) + I$. Since I is $\hat{\alpha}$ -stable, α is a monomorphism.

If y_{ij} denotes the image of \hat{y}_{ij} in R , then y_{11} is regular. But $\alpha(y_{11}) = y_{22}$ which is annihilated by y_{21} . Thus $\alpha(C(0)) \not\subseteq C(0)$.

Notice that by [16], theorem 7.2(ii), R is a commutative ring which does not have a Noetherian quotient ring.

5.18 Example:

The following is an example of a commutative domain R , Noetherian and of Krull dimension 1, with a monomorphism $\alpha: R \rightarrow R$ such that the

ring $A(R, \alpha)$ is a commutative domain, every factor ring of $A(R, \alpha)$ is uniform, and yet $A(R, \alpha)$ does not have Krull dimension.

Let R be the ring of formal power series $K[[y]]$ in one indeterminate over a field K , and let $\alpha: R \rightarrow R$ be the K -monomorphism such that $\alpha(y) = y^2$.

By corollary 2, p.131 of [22] (Vol. II), the only proper ideals of $K[[y]]$ are those of the form $\langle y^n \rangle$ (i.e. the ideal generated by y^n) for $n \in \mathbb{N}$. - (1).

Thus, R is a commutative, Noetherian domain of Krull dimension 1.

By corollary 1.26 and proposition 1.28, $A(R, \alpha)$ is a commutative domain.

Let I be a proper ideal of $A(R, \alpha)$, and assume if possible that there exists ideals $J, K \not\supseteq I$ such that the sum $K/I + J/I$ is direct. - (2).

Denote the α -sequences $r(K)$, $r(J)$ and $r(I)$ by $(K_i)_{i \geq 0}$, $(J_i)_{i \geq 0}$, and $(I_i)_{i \geq 0}$ respectively.

Then, as in the proof of lemma 4.10, the sum $K_i/I_i + J_i/I_i$ is direct, for all $i \geq 0$.

Since $I \neq K$, there exists $j \geq 0$, $r \in R$ such that $x^{-j} r x^j \in K - I$, i.e. $r \in K_j - I_j$. Similarly, since $I \neq J$ there exists $l \geq 0$, $s \in R$ with $s \in J_l - I_l$.

Then, $\alpha^l(r) \in \alpha^l(K_j) \subseteq K_{j+l}$, but $\alpha^l(r) \notin I_{j+l}$, otherwise $r \in I_j$, since $(I_i)_{i \geq 0}$ is an α -sequence. Hence $\alpha^l(r) \in K_{j+l} - I_{j+l}$ and similarly, $\alpha^j(s) \in J_{j+l} - I_{j+l}$.

Therefore $\frac{K_{j+l}}{I_{j+l}} \oplus \frac{J_{j+l}}{I_{j+l}}$ is a direct sum of two non-zero

submodules of R/I_{j+l} . But from (1) above, R/I is uniform. Thus, no direct sum such as (2) can exist, i.e. $\frac{A(R, \alpha)}{I}$ is uniform, for any proper ideal I of $A(R, \alpha)$.

Now, consider $\langle y^n \rangle$ where n is even. Clearly, $\langle y^{n/2} \rangle \subseteq \alpha^{-1}(\langle y^n \rangle)$. Furthermore, if $I \subseteq \langle y^{n/2+1} \rangle$ then $y^{n/2} \notin I$, and since $y^{n/2} \in \alpha^{-1}(\langle y^n \rangle)$, $I \neq \alpha^{-1}(\langle y^n \rangle)$.

Also, if $I \supseteq \langle y^{n/2-1} \rangle$ then $y^{n/2-1} \in I$ and $\alpha(y^{n/2-1}) = y^{n-2} \notin \langle y^n \rangle$. Thus $I \not\subseteq \alpha^{-1}(\langle y^n \rangle)$.

By (1) then, $\alpha^{-1}(\langle y^n \rangle) = \langle y^{n/2} \rangle$ for n even.

On the other hand, if n is odd, it is clear that $\langle y^{n+1/2} \rangle \subseteq \alpha^{-1}(\langle y^n \rangle)$, and a similar argument shows that $\alpha^{-1}(\langle y^n \rangle) = \langle y^{n+1/2} \rangle$.

$$\text{Thus, } \alpha^{-1}(\langle y^n \rangle) = \begin{cases} \langle y^{n/2} \rangle & \text{for } n \text{ even} \\ \langle y^{n+1/2} \rangle & \text{for } n \text{ odd} \end{cases}$$

Now, the argument of example 3.16 may be followed precisely, to show that $A(R, \alpha)$ does not have Krull dimension.

Further Questions.

- (1) Given that R is Noetherian, is it possible to define the Artinian radical in $A(R, \alpha)$? In other words, does the sum of all the Artinian left ideals of $A(R, \alpha)$ form an Artinian ideal? The problem here seems to be that, without $A(R, \alpha)$ being left Noetherian, it is difficult to see where the Artinian left ideals of $A(R, \alpha)$ come from - this is because the Artinian left ideals no longer necessarily coincide with the left ideals which have finite length as $A(R, \alpha)$ -modules.

If the Artinian radical of $A(R, \alpha)$ could successfully be defined, then it may be possible to obtain a Stafford-like result superior to theorem 4.6.

Also it could happen that the theorem of Ginn and Moss ([3], theorem 4.14) which says that if R is a Noetherian order in an Artinian ring, then the Artinian radical is a direct summand of R , works in $A(R, \alpha)$ without assuming that $A(R, \alpha)$ is Noetherian. The fact (corollary 7.3 of [16]) that if R is a left order in a left Artinian ring then so is $A(R, \alpha)$, is encouraging here.

- (2) If R is a left Goldie ring, is $A(R, \alpha)$ a left Goldie ring? Example 2.25 shows that the ascending chain condition for left annihilators can be lost on passage from R to $A(R, \alpha)$, but in this example, R has infinite Goldie dimension.

- (3) If R is left Noetherian, is the nilpotent radical $N(R)$ α -invariant? Indeed, is the Jacobson radical $J(R)$ α -invariant? The result obtained in theorem 3.34 appears to indicate that $J(R)$ may be more relevant than $N(R)$ in any attempt at generalizing Jategaonkar's result (lemma 1.1 of [13]) for left Artinian rings. It also may indicate that chain conditions in $A(R, \alpha)$ have more bearing than chain conditions in R .
- (4) In theorems 5.7 and 5.10, two logically independent criteria are found for the skew Laurent polynomial ring $R[x, x^{-1}, \alpha]$ to be primitive. Two very similar conditions, also logically independent, are known to be sufficient for the Ore extension $R[x, \delta]$ to be primitive, where $\delta: R \rightarrow R$ is a derivation. (See theorems 1 and 2 of [14]).

In their recent paper [8], Goodearl and Warfield show that if $R[x, \delta]$ is primitive, then one of the above mentioned conditions on R must hold. This is done by exhibiting two types of faithful, irreducible $R[x, \delta]$ -modules, each of which corresponds to one of the conditions on R , and then showing that these are the only types possible. In view of the striking similarity of primitivity-type results for $R[x, \delta]$ and $R[x, x^{-1}, \alpha]$, (compare [14] and [15]), it is possible that this approach may work for $R[x, x^{-1}, \alpha]$, where $\alpha: R \rightarrow R$ is an automorphism. The theory of $A(R, \alpha)$ could then be

used to extend this to the case where $\alpha: R \rightarrow R$ is a monomorphism.

Goodearl and Warfield [9] have studied the Krull dimension of $R[x, \delta]$, and this may also be useful if applied to $R[x, x^{-1}, \alpha]$.

- (5) Homological properties for $A(R, \alpha)$ have hardly been mentioned at all. Fields [5] has shown that if $\alpha: R \rightarrow R$ is a monomorphism, then the left global dimension of $R[x, \alpha]$ cannot exceed $1 + \text{lgd}(R)$, $\text{lgd}(R)$ being the left global dimension of R .

It would be interesting to aim for some similar result in $A(R, \alpha)$.

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